Precoding Techniques for Undersampled Multireceiver Communication Systems

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***Abstract*—Although there have been extensive studies on the blind equalization problem for temporally or/and spatially oversampled communication signals, the literature is not equally rich for undersampled, and possibly multiuser, wireless systems. In this paper, we investigate the feasibility of blind signal recovery from undersampled data collected from plurality receivers. We show that although an undersampled communication system is not completely identifiable in general, such an obstacle can be overcome by employing proper precoding with an arbitrary amount of bandwidth expansion in the transmitter. The main contribution of this paper is the formulation of a generic framework for all undersampled systems and the derivation of necessary and sufficient conditions for a class of filters (which we term *ambiguity resistant* precoders) that allow the blind recovery of the transmitted signal. A family of such precoders is given. We discuss various related issues and show, for example, that block precoders, i.e., memoryless precoders, are not ambiguity resistant. In addition, we derive an algorithm that can accomplish blind equalization with a finite number of observations.**

***Index Terms*—Antenna array, blind estimation, precoding.**

I. INTRODUCTION AND PROBLEM FORMULATION

BECAUSE of its practical significance, blind identification of an FIR linear time invariant (LTI) channel has received considerable attention in the past decade in communications and signal processing [1], [2]. There are two basic issues in blind identification—one on blind identifiability and the other on the identification methodology when a system is blindly identifiable. In this paper, we mainly focus on the first issue, i.e., the blind identifiability.

An LTI channel can be described as

$$y(t) = \sum_n s[n]h(t - nT) \tag{1}$$

where

$s[n]$	information bearing symbol sequence with baud rate $1/T$;
$h(t)$	LTI channel;
$y(t)$	received signal.

In blind identification, the goal is to determine $s[n]$ from the received signal $y(t)$ without knowledge of the channel $h(t)$. It is known that when $y(t)$ is sampled at the baud rate $1/T$, such a task is not possible without the use of higher order statistics unless the channel is specially structured (e.g., a minimum-phase channel). The situation is completely different when the channel outputs possess certain diversities. Tong *et al.* show that almost all FIR channels can be blindly determined if the channel output is oversampled by means of fractionally sampling or/and multiple receivers [3], [4].

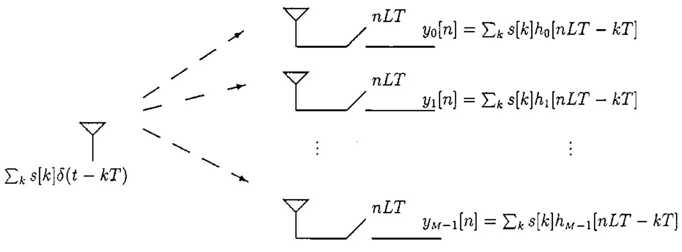

Fig. 1. Undersampled antenna array system.

The basic principle of diversity techniques in blind identification is the following. Let $S(z)$, $H_1(z)$, and $H_2(z)$ be three polynomials of z. When more than one output $Y_1(z) = H_1(z)S(z)$ and $Y_2(z) = H_2(z)S(z)$ are available, all zeros of $S(z)$ (and $S(z)$ itself) can be determined from the zeroes of $Y_1(z)$ and $Y_2(z)$ if and only if $H_1(z)$ and $H_2(z)$ are coprime, i.e., they do not have any common zeros. For a fractionally sampled system with oversampling rate 2, i.e., $N = 2$, $H_1(z)$, and $H_2(z)$ correspond to the two polyphase components of the channel $H(z)$. For a multireceiver system with tworeceivers, $H_1(z)$ and $H_2(z)$ correspond to the two channel transfer functions of the two antennas. The two types of diversities can be combined, and there is an abundance of references in the literature on these problems; see, for example, [5]–[11].

To date, almost all research on blind identification deals with channel outputs that are sampled at least at the baud rate. In certain applications, a communication system may be *undersampled* with rate $1/LT$ $(L > 1)$ for reasons ranging from fixed hardware to variable data rates of source signals. Clearly, perfect signal recovery is impossible in these scenarios. However, when a collection of low rate observations is available, it may be feasible to restore the source signals by combining partial information from different receivers. In this paper, we study the application of multiple receivers in blind source recovery for undersampled communication systems. To put this into perspective, consider an M-receiver undersampled system depicted in Fig. 1, where L is an integer. The question herein is whether or not we can identify $s[n]$ from the received signals $\{y_m[n]\}_{m=0}^{M-1}$. At a first glance, one may be tempted to reply "yes" because the system appears to have sufficient redundancy when $M > L$.

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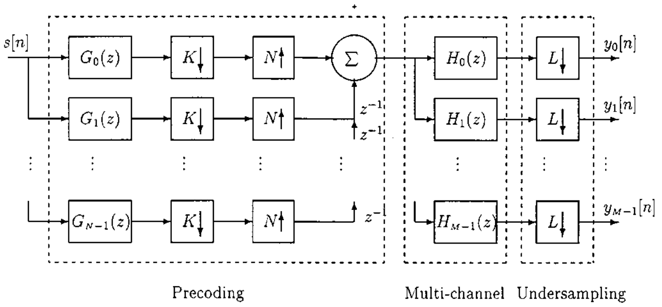


Fig. 2. Precoding for undersampled an antenna array system.

The correct answer, nevertheless, is no when $L > 1$. This is because the system in Fig. 1, as will be shown in the ensuing sections, is equivalent to a multiple input and multiple output (MIMO) system. Based on the previous known results [8], [12], [13], one can at most identify an MIMO system to a matrix ambiguity. The question, then, becomes the following: Is there any affordable way to restore the blind identifiability? To answer this question, let us briefly recall a precoding scheme recently introduced in [14] and [15] for intersymbol interference (ISI) cancellation using nonmaximally decimated multirate filterbanks.

The precoding scheme proposed in [14] and [15] can be shown in the left-most block in Fig. 2, where $K\downarrow$ means downsampling by a factor of K, i.e., taking only one sample in each K samples, and $N\uparrow$ stands for upsampling by a factor N, i.e., inserting $N-1$ zeroes between each two adjacent samples. It has been shown in [14] that almost all FIR LTI channels $H(z)$ can be ideally equalized using FIR precoding filters with an expansion in bandwidth: $1/N$, i.e., $K = N - 1$. The same concept has been applied by Giannakis to blind channel identification in [16]. Precoding can also to used to shape the spectra and introduce spectral asymmetry between multiple sources. A method based on correlative precoding or partial-response signals has been proposed by Xavier *et al.* for blind multiuser channel identification in spatial division multiple access (SDMA) applications [18]. Other ISI related precoding techniques, e.g., Tomlinson-Harashima (TH) coding [19], [20] improve the performance of a communication link but require explicit knowledge of the channel characterisitcs.

Motivated by the above observations, we propose to use precoding techniques to solve the blind identification problem for the undersampled system in Fig. 1. The overall structure is mathematically described by Fig. 2 without loss of generality. The rationale here is that although it is not possible to completely identify $s[n]$ in Fig. 1, one should be able to restore the identifiability by slightly adding some *manageable* redundancies in the transmitter. By comparing the new system in Fig. 2 with the previous system in Fig. 1, it is seen that the samples at each receiver are reduced by L times, whereas the samples at the transmitter are increased by N/K times. Therefore, the overall samples are reduced by $L(K/N) = LK/N$ times. In particular, when $L = N$, the overall samples in the new system in Fig. 2 are reduced by K times. When K is chosen to be as large as $N-1$ with $N = L \leq M$ for M receivers, this reduction of samples might have significant impact on the computational load at each receiver. In addition, the undersampling reduces the sampling rate at the receiver, which may reduce the hardware cost significantly in high-speed digital communication systems.

The primary issue we address in this paper is the precoder conditions under which the input signal $s[n]$ can be blindly determined from the received signals $\{y_m[n]\}_{m=0}^{M-1}$ without knowledge of the channel characteristics $\{H_m(z)\}_{m=0}^{M-1}$. More specifically, we study a class of *ambiguity resistant* precoders that is capable of removing the ambiguity introduced by undersampling. In the remainder of this paper, we will denote the system in Fig. 2 as $[(K,N);(L,M)]$. This identifiability question will be investigated under a unified framework developed in Section II, where we show that all $[(K,N);(L,M)]$ systems can be cast into a $[(\underline{K},\underline{N});(\underline{N},\underline{M})]$ system with $\underline{K} < \underline{N} < \underline{M}$, provided that $N > K$ and $MN > KL$. In Section III, we derive necessary and sufficient conditions on ambiguity resistant precoders in blind identification. A family of such precoders is also given. Surprisingly, we show that all block precoders, i.e., memoryless precoders, are not ambiguity resistant unless $K = 1$. Although the emphasis of this study is not on algorithm development, we nevertheless present a data-efficient algorithm that can accomplish blind source recovery for a $[(K,N);(N,M)]$ system with simple least-squares fitting. The algorithm is evaluated by numerical simulations in Section V, and the paper is concluded in Section VI.

II. SYSTEM UNIFICATION

In this section, some fundamentals of filterbank operations are first reviewed. We then formulate the system of interest using an MIMO model, based on a generic framework we introduce that will be used throughout this paper. Although our problem is introduced in the context of undersampled multireceiver systems, it will become clear in the ensuring sections that the results presented have much broad applications.

As a general notational convention, matrices (in capital letters) and vectors (in low cases) will be in boldface. The symbols $(\cdot)^H$, $(\cdot)^T$, and $\otimes$ stand for Hermitian, transpose, and convolution, respectively; $\mathbf{H}(z) = \mathbf{H}_0 + \mathbf{H}_1 z^{-1} + \cdots + \mathbf{H}_p z^{-p}$ denotes a matrix polynomial. The symbol $\mathbf{I}(\mathbf{0})$ stands for the identity (zero) matrix or vector with a proper dimension. If $\mathbf{H}_p \neq \mathbf{0}$, then p is called the *order* of the polynomial matrix $\mathbf{H}(z)$.

A. MIMO Representation

For analytical simplicity, it is more convenient to represent the system in Fig. 2 as an MIMO system with matrix transfer functions.

Since the (un)blocking process and the polyphase representation will play an important role in the rest of this paper, let us define some notations and review some basic results; see [21] and [22] for related topics.

The output $\mathbf{y}[n]$ shown in Fig. 3(a) of the blocked $y[n]$ with block size N is the vector-valued signal $\mathbf{y}[n] = (y[Nn], y[Nn-1], \ldots, y[Nn-N+1])^T$. Conversely, the output $w[n]$ shown in Fig. 3(b) of the unblocked vector-valued signal $\mathbf{w}[n] = [w_0[n], w_1[n], \ldots, w_{N-1}[n])^T$ with unblock size N is $w[n] = w_k[l]$ when $n = Nl - k$ for $k = 0, 1, \ldots, N-1$. In particular, when $\mathbf{w}[n] = (y[Nn], y[Nn-1], \ldots, y[Nn-N+1])^T$, then $w[n] = y[n]$.

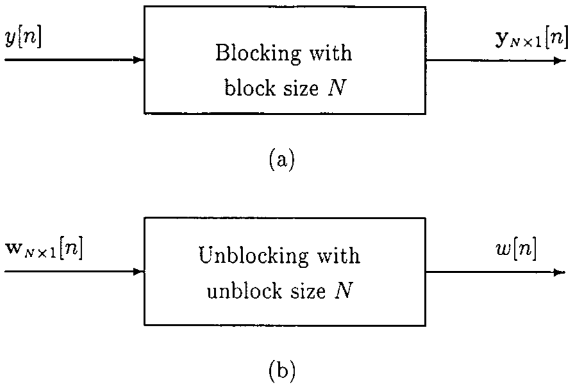


Fig. 3. Blocking and unblocking.

The same blocking and unblocking processes can be similarly defined for vector or matrix sequences.

For a filter $Q(z) = \sum_n q[n]z^{-n}$, its γth *polyphase component* of size Γ, $Q_\gamma(z)$ is defined as

$$Q_\gamma(z) = \sum_n q[\Gamma n + \gamma]z^{-n}, \quad 0 \le \gamma \le \Gamma - 1. \tag{2}$$

For the M channels responses in Fig. 2 $\{H_m(z)\}_{m=0}^{M-1}$, their *polyphase matrix* with size $M \times L$ is denoted by

$$\mathbf{H}(z) = (H_{m,l}(z))_{0\le m\le M-1, 0\le l\le L-1} \tag{3}$$

where $H_{m,l}(z)$ is the lth polyphase component of size L of $H_m(z)$. Similarly, for the precoders $\{G_n(z)\}_{n=0}^{N-1}$, their polyphase matrix $\mathbf{G}(z)$ $(N \times K)$ is given by

$$\mathbf{G}(z) = (G_{n,k}(z))_{0\le n\le N-1, 0\le k\le K-1} \tag{4}$$

where $G_{n,k}(z)$ is the kth polyphase component of size K of $G_n(z)$.

With the above notations, the system in Fig. 2 can be converted to an equivalent MIMO system shown in Fig. 4 [14], [21].

B. Generic Framework

Throughout our discussion, the follow assumptions are invoked for the $[(K,N);(L,M)]$ system under consideration.

A.1) The precoding filter has dimension $N \times K$, where $N > K$;
A.2) $N/K \times M/L > 1$, i.e., $NM > KL$.

A.1 is clearly required in the precoding; otherwise, there will be no increase in redundancy that renders blind identification impossible. The same is true for A.2. MN/KL quantifies the overall system redundancy. If $NM \le KL$, the whole system becomes an MIMO system with no more outputs than inputs, which is not possible to be identified blindly.

Under A.1 and A.2, there are still many possible combinations of the four parameters (K, N, L, and M), which make a unified analysis difficult. The following lemma simplifies our data model by casting any system that satisfies A.1 and A.2 into a generic framework.

Lemma 1: Any $[(\underline{K},\underline{N});(\underline{L},\underline{M})]$ system with $\underline{N} > \underline{K}$ and $\underline{NM} > \underline{KL}$ can be cast into a generic $[(K,N);(N,M)]$ system with $K < N < M$.

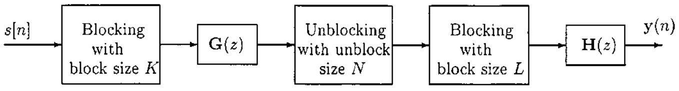


Fig. 4. Equivalent precoded undersampled antenna array system.

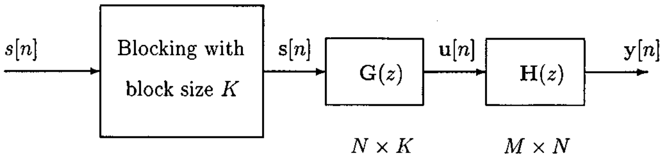


Fig. 5. Generic undersampled multireceiver system with precoding filters.

Proof: See Appendix A. □

Lemma 1 enables us to limit our analysis to the $\{[(K,N);(N,M)] : K < N < M\}$ system depicted in Fig. 5 without loss of generality.

III. IDENTIFIABILITY ANALYSIS

The output of the generic system in Fig. 5 can be expressed as

$$\begin{aligned}\mathbf{y}_{M\times 1}(z) &= \mathbf{H}_{M\times N}(z)\mathbf{u}_{N\times 1}(z)\\ &= \mathbf{H}_{M\times N}(z)\mathbf{G}_{N\times K}(z)\mathbf{s}_{K\times 1}(z)\end{aligned}$$

where $\mathbf{H}(z)$ characterizes the *unknown* channel, whereas $\mathbf{G}(z)$ represents the *known* precoder. *The problem herein is to determine the input* $\mathbf{s}(z)$ *and in many cases the channel* $\mathbf{H}(z)$ *from the output* $\mathbf{y}(z)$ *using only knowledge of the precoder filter* $\mathbf{G}(z)$.

This section addresses the conditions for $\mathbf{H}(z)$ and $\mathbf{G}(z)$ under which $\mathbf{s}_{K\times 1}(z)$ can be blindly identified from $\mathbf{y}_{M\times 1}(z)$. To facilitate the forthcoming discussion, let us first lay some groundwork by reviewing an important result regarding FIR MIMO systems.

Theorem 1 [12], [23], [24]: For an N-input M-output $(M > N)$ FIR system with transfer function $\mathbf{H}(z)$, the following statements are equivalent:

1) $\mathbf{H}(z)$ is irreducible, i.e., rank$[\mathbf{H}(z)] = N$, $\forall z \in \mathcal{C}$, and rank$[\mathbf{H}_0] = N$ [25].
2) $\mathbf{H}(z)$ and the input vector $\mathbf{u}(z)$ can be identified up to an $N \times N$ invertible constant ambiguity matrix from the outputs using second-order statistics.

Most practical channels are irreducible. Extreme cases (e.g., instantaneous channels in narrowband antenna array systems) can be coped with receiver-end processing that removes the coherence of the output signals. If the precoder is also designed to be irreducible, the *composite* transfer function $\mathbf{H}_c(z) \overset{\text{def}}{=} \mathbf{H}(z)\mathbf{G}(z)$ is clearly irreducible. Theorem 1 asserts that the system input $\mathbf{s}(z)$ can *only* be determined within a $K \times K$ matrix ambiguity directly from $\mathbf{y}(z)$ if the precoder $\mathbf{G}(z)$ and the channel $\mathbf{H}(z)$ are combined as a single MIMO system. Based on this observation, it seems impossible to fully recover the input since any combination of $\tilde{\mathbf{s}}(z) = \mathbf{W}\mathbf{s}(z)$ and $\tilde{\mathbf{H}}_c(z) = \mathbf{H}_c(z)\mathbf{W}^{-1}$ can qualify

$$\tilde{\mathbf{H}}_c(z)\tilde{\mathbf{s}}(z) = \mathbf{y}(z).$$

While the above equality holds indeed for many possible inputs, the conclusion regarding the identifiability is invalid. This is because that the above approach clearly does not take any advantage of the precoding part. Instead of finding an $\mathbf{s}(z)$ and $\mathbf{H}_c(z)$ that satisfy $\mathbf{y}(z) = \mathbf{H}_c(z)\mathbf{s}(z)$, the problem of interest here is to find $\mathbf{s}(z)$ and $\mathbf{H}_c(z)$ such that

$$\mathbf{y}(z) = \mathbf{H}_c(z)\mathbf{s}(z), \quad \text{subject to } \mathbf{H}_c(z) = \mathbf{H}(z)\mathbf{G}(z)$$

where $\mathbf{G}(z)$ is a known precoder. This motivates the following blind identifiability concept.

Definition: The system in Fig. 5 is blindly identifiable if $\tilde{\mathbf{s}}(z) = \alpha\mathbf{s}(z)$ and $\tilde{\mathbf{H}}(z) = \beta\mathbf{H}(z)$, where α and β are two scalars, are the only solution for the following system given the output $\mathbf{y}(z)$ and the precoder $\mathbf{G}(z)$:

$$\mathbf{y}(z) = \tilde{\mathbf{H}}(z)\mathbf{G}(z)\tilde{\mathbf{s}}(z).$$

In the following, we show that by incorporating knowledge of the precoder, it is possible to determine the input $\mathbf{s}(z)$ within a scalar factor. We tackle the blind identification problem in two steps. i) Determine what we term the *ambiguous inputs*

$$\tilde{\mathbf{u}}(z) : \mathbf{T}\tilde{\mathbf{u}}(z) = \mathbf{u}(z) \tag{5}$$

where $\mathbf{T}$ is an $N \times N$ full-rank constant matrix, blindly from the system output $\mathbf{y}(z)$. This can be accomplished using many existing approaches when $\mathbf{H}(z)$ is irreducible, provided that the inputs have sufficient linear complexity. ii) Once $\tilde{\mathbf{u}}(z)$ is identified, the blind identification problem reduces to whether or not $\mathbf{s}(z)$ can be determined from $\tilde{\mathbf{u}}(z)$ in the presence of a full-rank ambiguity matrix $\mathbf{T}$ (or $\mathbf{T}^{-1}$). We then show that there exists a class of *ambiguity resistant* precoders that can resolve the matrix ambiguity without additional information. Since Step i) is well studied [8], [12], [13], our focus in the remainder of this paper will be devoted to the precoder part.

A. Ambiguity Resistant (AR) Precoders

We first define the concept of ambiguity resistance.

Definition: An $N \times K$ FIR irreducible precoding filter $\mathbf{G}(z)$ is *ambiguity resistant* if its input $\mathbf{s}(z)$ can be uniquely determined up to a scalar from its ambiguous output $\{\tilde{\mathbf{u}}(z) : \mathbf{T}\tilde{\mathbf{u}}(z) = \mathbf{u}(z)\}$, where $\mathbf{T}$ is an unknown invertible $N \times N$ constant matrix.

The above definition asserts that given $\tilde{\mathbf{u}}(z)$ and a known ambiguity resistant precoder $\mathbf{G}(z)$, the unknown pair $(\mathbf{T}, \mathbf{s}(z))$ can be uniquely determined up to a scalar ambiguity based on the following relationship:

$$\mathbf{T}\tilde{\mathbf{u}}(z) = \mathbf{u}(z) = \mathbf{G}(z)\mathbf{s}(z). \tag{6}$$

Although the above definition is intuitive, it is hard to handle in practice. We next want to simplify the ambiguity resistance and propose another condition that is simpler to study. Note that if there exists an $N \times N$ full-rank, nonidentical, constant matrix $\mathbf{E}$ and a $K \times K$ nonidentical matrix $\mathbf{X}(z)$ such that

$$\mathbf{E}\mathbf{G}(z) = \mathbf{G}(z)\mathbf{X}(z) \tag{7}$$

it is straightforward to show that $(\mathbf{E}\mathbf{T}, \mathbf{X}(z)\mathbf{s}(z))$ also satisfied (6).

Defining $\tilde{\mathbf{s}}(z) = \mathbf{X}(z)\mathbf{s}(z)$, it is clear that $\mathbf{X}(z)$ is the *polynomial ambiguity* of the input that cannot be determined.

The above is summarized in the following theorem.

Theorem 2: An $N \times K$ FIR irreducible precoding filter $\mathbf{G}(z)$ is ambiguity resistant if there does not exist an $N \times N$ full-rank constant matrix $\mathbf{E} \neq \alpha\mathbf{I}$ for any constant α and a $K \times K$ matrix $\mathbf{X}(z) \neq \beta\mathbf{I}$ for any constant β such that the above identity (7) holds.

Proof: Arrange K segments of the input signals, where each segement is a $K \times 1$ polynomial vector $\mathbf{s}_k(z)$ into a $K \times K$ input matrix $\mathbf{S}(z) = [\mathbf{s}_1(z) \;\cdots\; \mathbf{s}_K(z)]$. Since the input signal is random, this $K \times K$ matrix almost surely has full rank. Let $(\mathbf{T}, \tilde{\mathbf{S}}(z))$ be another set of solution that satisfies

$$\mathbf{T}\mathbf{G}(z)\tilde{\mathbf{S}}(z) = \mathbf{G}(z)\mathbf{S}(z).$$

To prove this theorem, we only need to show that under the condition in the theorem, we can conclude that $\tilde{\mathbf{S}}(z) = \alpha\mathbf{S}(z)$ for a nonzero constant α. Since $\mathbf{T}$ is invertible, the above equation implies

$$\mathbf{T}^{-1}\mathbf{G}(z) = \mathbf{G}(z)\tilde{\mathbf{S}}(z)\mathbf{S}^{-1}(z).$$

By the condition in the theorem, we have $\tilde{\mathbf{S}}(z)\mathbf{S}^{-1}(z) = \alpha\mathbf{I}_K$, which proves the theorem. ☐

The uniqueness of the solution $(\mathbf{T}, \mathbf{s}(z))$ in (6) tells us that

$$\mathbf{T}\mathbf{G}(z)\tilde{\mathbf{s}}(z) = \mathbf{G}(z)\mathbf{s}(z)$$

implies that $\tilde{\mathbf{s}}(z) = \alpha\mathbf{s}(z)$. In other words, the solution of $\mathbf{X}(z)$ has to be $\alpha\mathbf{I}$ in (7), whereas constant matrix $\mathbf{T}$ does not have to be an identical matrix. A more detailed analysis of these conditions has been discussed in our follow-up work [17]. For simplicity, in what follows, we only consider the sufficient condition in Theorem 2 for ambiguity resistant precoders, i.e., $\mathbf{G}(z)$ is called ambiguity resistant if (7) in terms of variables in $\mathbf{E}$ and $\mathbf{X}(z)$ has only the trivial solution $\mathbf{E} = \alpha\mathbf{I}$ and $\mathbf{X}(z) = \alpha\mathbf{I}$ for a constant α. Note that since $\mathbf{G}(z)$ is irreducible, $\det(\mathbf{X}(z))$ in (7) is a nonzero constant, i.e., $\mathbf{X}(z)$ is unimodular.

To examine this ambiguity resistancy of a given precoder, one would need to determine if (7) has a unique solution, i.e., whether $\mathbf{X}(z) = \mathbf{I}$ (which guarantees $\tilde{\mathbf{s}}(z) = \mathbf{s}(z)$) is the only solution. To make the precedure numerically tractable, note that it follows from (7) that $\mathbf{X}(z) = \mathbf{G}_L(z)\mathbf{E}\mathbf{G}(z)$, where $\mathbf{G}_L(z)$ denotes the left pseudo-inverse of $\mathbf{G}(z)$. Hence

$$\mathbf{E}\mathbf{G}(z) = \mathbf{G}(z)\mathbf{G}_L(z)\mathbf{E}\mathbf{G}(z). \tag{8}$$

By representing the above equation in the time domain, one may check the ambiguity resistancy of $\mathbf{G}(z)$ by solving a linear equation set. If $\mathbf{E} = \alpha\mathbf{I}$ for some constant α as the only nonzero solution, then $\mathbf{G}(z)$ is ambiguity resistant. Otherwise, it is necessary to check whether $\mathbf{X}(z) = \beta\mathbf{I}$ for some constant β or $\mathbf{E}\mathbf{G}(z) = \mathbf{G}(z)$ since it is possible to have $\mathbf{E}\mathbf{G}(z) = \mathbf{G}(z)$ with $\mathbf{E} \neq \alpha\mathbf{I}$.

When $K = 1$, $\mathbf{X}(z) = \alpha$ for some nonzero constant α is always true. By Theorem 2, the following corollary is straightforward.

Corollary 1: An $N \times 1$ FIR invertible precoding filter $\mathbf{G}(z)$ is always ambiguity resistant for $N > 1$.

This corollary is not surprising since when $K = 1$, the $[(1, N); (N, M)]$ system reduces to a conventional oversampled system that is clearly identifiable. With this result, we only need to consider the case of $K > 1$.

Next, we want to present some necessary conditions on the ambiguity resistance.

Theorem 3: If an $N \times K$, $K > 1$, FIR irreducible precoder $\mathbf{G}(z)$ is ambiguity resistant, then we have the following.

1) There exist no full-rank constant matrix $\mathbf{E}$ and invertible $K \times K$ polynomial matrix $\mathbf{V}(z)$ such that the first column in the matrix $\mathbf{E}\mathbf{G}(z)\mathbf{V}(z)$ is $(1, 0, 0, \ldots, 0)^T$.
2) $N > K$.
3) The order Q of $\mathbf{G}(z)$ must satisfy the following lower bound:

$$Q \geq \frac{N^2 + K^2 - 1}{NK} - 1.$$

Proof: See Appendix B. □

Condition 2 is not surprising since when $N \leq K$, no redundancies are introduced, and therefore, the precoder does not have any ambiguity resistant capability. As a side note on Condition 3, it can be easily verified that when $K = N - 1$, $Q = 1$ marginally satisfies the above condition. Therefore, a first-order precoder could be sufficient for an $[(N-1, N); (N, M)]$ system. We will see an example of such cases later. These necessary conditions are helpful in the construction of ambiguity resistant precoders $\mathbf{G}(z)$ later. From the above necessary Condition 1, we immediately have the following corollary.

Corollary 2: Block precoders $\mathbf{G}(z)$ (constant matrix) with $K > 1$ are not ambiguity resistant.

Proof: When $\mathbf{G}(z)$ itself is a constant matrix, there always exists an $N \times N$ full-rank and nonidentical constant matrix $\mathbf{E}$ such that the first column of the matrix $\mathbf{E}\mathbf{G}(z)$ is $(1, 0, 0, \ldots, 0)^T$. □

Although these three necessary conditions in the above theorem are rich enough in constructing examples (see later) of ambiguity resistant matrices, they are still not sufficient. A counterexample is the following 4×2 polynomial matrix:

$$\mathbf{G}(z) = \begin{bmatrix} 1 & 0 \\ z^{-\gamma} & 0 \\ 0 & 1 \\ 0 & z^{-\gamma} \end{bmatrix}$$

where $\gamma > 1$ is a fixed integer. It is not hard to see that this $\mathbf{G}(z)$ satisfies the above all three necessary conditions when γ is large enough. It is, however, not ambiguity resistant, which can be seen from the following lemma as a general result.

Lemma 2: Let $\mathbf{G}_j(z)$ be $N_j \times K_j$ polynomial matrices for $j = 1, 2, \ldots, J > 1$. Then, the block diagonal polynomial matrix $\text{diag}(\mathbf{G}_1(z), \mathbf{G}_2(z), \ldots, \mathbf{G}_J(z))$ is not ambiguity resistant.

Proof: Clearly, $\alpha_j \mathbf{I}_{N_j \times N_j} \mathbf{G}_j(z) = \mathbf{G}_j(z) \alpha_j \mathbf{I}_{K_j \times K_j}$ for $j = 1, 2, \ldots, J$. Let $\mathbf{E} = \text{diag}(\alpha_1 \mathbf{I}_1, \alpha_2 \mathbf{I}_2, \ldots, \alpha_J \mathbf{I}_J)$ and $\mathbf{V}(z) = \text{diag}(\alpha_1 \mathbf{I}_1, \alpha_2 \mathbf{I}_2, \ldots, \alpha_J \mathbf{I}_J)$. Then

$$\begin{aligned} &\mathbf{E}\,\text{diag}(\mathbf{G}_1(z), \mathbf{G}_2(z), \ldots, \mathbf{G}_J(z)) \\ &\quad = \text{diag}(\mathbf{G}_1(z), \mathbf{G}_2(z), \ldots, \mathbf{G}_J(z))\mathbf{V}(z). \end{aligned}$$

As long as $\alpha_1, \alpha_2, \ldots, \alpha_J$ are not all equal, $\mathbf{E} \neq \alpha\mathbf{I}$ and $\mathbf{V}(z) \neq \beta\mathbf{I}$ for any constants α and β. This proves the lemma. □

With the establishment of the above necessary conditions, we now present a theorem concerning the necessary and sufficient conditions for an ambiguity resistant precoder. Although the physical interpretation of our results is unclear at this stage, the theorem is nevertheless interesting from a theoretical viewpoint. The proof of the theorem builds on the following two useful lemmas.

Lemma 3 [21]: When $\mathbf{G}_{N \times K}(z)$ $(N > K)$ is irreducible, its Smith–McMillan form is given by

$$\mathbf{G} = \mathbf{W}(z) \begin{bmatrix} \mathbf{I}_{K \times K} \\ \mathbf{0} \end{bmatrix} \mathbf{V}(z)$$

where $\mathbf{W}_{N \times N}(z)$ and $\mathbf{V}_{K \times K}(z)$ are referred to as the left and right unimodular matrices, respectively, in the Smith–McMillan decomposition of $\mathbf{G}(z)$.

The left unimodular matrix $\mathbf{W}(z)$ can be further decomposed into

$$\mathbf{W}(z) = \begin{bmatrix} \mathbf{W}^s_{N \times K}(z) & \mathbf{W}^c_{N \times (N-K)}(z) \end{bmatrix}.$$

Clearly, $\mathbf{W}^s(z)$, which is associated with the identity part of the middle Smith form, essentially defines the column span of $\mathbf{G}(z)$. The Smith–McMillan decomposition of a tall invertible matrix can be simplied as $\mathbf{G}(z) = \mathbf{W}^s(z)\mathbf{V}(z)$, where $\mathbf{W}^s(z)$ is left invertible.

Lemma 4: Let $\mathbf{G}(z) = \mathbf{W}_1(z)[\mathbf{I}\ \mathbf{0}]^T\mathbf{V}_1(z) = \mathbf{W}_2(z)$ $[\mathbf{I}\ \mathbf{0}]^T\mathbf{V}_2(z)$ be two Smith–McMillan decompositions of the precoding filter. $\mathbf{V}_1(z) = \alpha\mathbf{V}_2(z)$ if and only if $\mathbf{W}^s_1(z) = 1/\alpha\mathbf{W}^s_2(z)$, where α is a nonzero scalar.

The proof is simple and is thus omitted.

Theorem 4: A precoding filter $\mathbf{G}(z)$ is ambiguity resistant if and only if there exists no $N \times N$ constant matrix $\mathbf{E}$ such that both $\mathbf{W}(z)$ and $\mathbf{E}\mathbf{W}(z)$, $\mathbf{W}^s(z) \neq \alpha\mathbf{E}\mathbf{W}^s(z)$ are left unimodular matrices in the Smith–McMillan decompositions of $\mathbf{G}(z)$.

Proof:

Necessary Part:

If $\mathbf{G}(z) = \mathbf{W}^s(z)\mathbf{V}(z) = \mathbf{E}\mathbf{W}^s(z)\mathbf{V}_2(z)$, post multiplying $\mathbf{V}_2^{-1}(z)\mathbf{V}(z)$ to all sides yields

$$\mathbf{W}^s(z)\mathbf{V}(z)\mathbf{V}_2^{-1}(z)\mathbf{V}(z) = \mathbf{E}\mathbf{W}^s(z)\mathbf{V}(z).$$

Let $\mathbf{X}(z) = \mathbf{V}_2^{-1}(z)\mathbf{V}(z)$. Since $\mathbf{W}^s(z) \neq \alpha\mathbf{E}\mathbf{W}^s(z)$, Lemma 4 asserts that $\mathbf{X}(z) \neq \alpha\mathbf{I}$. It immediately follows that

$$\mathbf{E}\mathbf{G}(z) = \mathbf{G}(z)\mathbf{X}(z).$$

The above equation implies the unimodularity of $\mathbf{X}(z)$. Therefore, $\mathbf{G}(z)$ is not ambiguity resistant based on Theorem 2.

Sufficient Part:

Let $\mathbf{G}(z) = \mathbf{W}(z)[\mathbf{I}\ \mathbf{0}]^T\mathbf{V}(z) = \mathbf{W}^s(z)\mathbf{V}(z)$. If there exists an $\mathbf{X}(z) \neq \alpha\mathbf{I}$ and

$$\mathbf{E}\mathbf{G}(z) = \mathbf{G}(z)\mathbf{X}(z)$$

then

$$\mathbf{E}\mathbf{W}^s(z)\mathbf{V}(z)\mathbf{X}^{-1}(z) = \mathbf{W}^s(z)\mathbf{V}(z) = \mathbf{G}(z).$$

Clearly, both $\mathbf{W}(z)$ and $\mathbf{E}\mathbf{W}(z)$ are valid left unimodular matrices of the Smith–McMillan decomposition. Since $\mathbf{V}(z)\mathbf{X}^{-1}(z) \neq 1/\alpha\mathbf{V}(z)$, Lemma 4 asserts that $\mathbf{E}\mathbf{W}^s(z) \neq \alpha\mathbf{W}^s(z)$. □

It would be very interesting to investigate simpler necessary and sufficient conditions on ambiguity resistant precoders.

B. Family of AR Precoders

Motivated from the necessary conditions in Theorem 3, we now want to construct a family of ambiguity resistant precoders $\mathbf{G}(z)$. We have the following result.

Theorem 5: For any positive integer $N > 1$, the following matrix $\mathbf{G}(z)$ with size $N \times (N-1)$ is ambiguity resistant:

$$\mathbf{G}(z) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ z^{-\gamma} & 1 & 0 & \cdots & 0 & 0 \\ 0 & z^{-\gamma} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z^{-\gamma} & 1 \\ 0 & 0 & 0 & \cdots & 0 & z^{-\gamma} \end{bmatrix}_{N\times(N-1)} \tag{9}$$

for an integer $\gamma \neq 0$.

Proof: To prove this theorem, it is sufficient to prove it for the case of $\gamma = 1$; otherwise, we may replace $z^{-\gamma}$ by a new variable $\hat{z}^{-1}$.

First of all, the above $\mathbf{G}(z)$ is irreducible. Define the first equation at the bottom of the page. It is easy to check that

$$\mathbf{G}_L(z)\mathbf{G}(z) = \mathbf{I}_{(N-1)\times(N-1)}.$$

Let $\mathbf{E} = (e_{m,n})_{N\times N}$ be an $N \times N$ constant matrix such that $\mathbf{EG}(z) = \mathbf{G}(z)\mathbf{V}(z)$. We want to prove that $\mathbf{E} = \alpha\mathbf{I}$ for some constant α.

By the above equations, we have $\mathbf{V}(z) = \mathbf{G}_L(z)\mathbf{EG}(z)$. It is clear that we then have the second equation at the bottom of the page. Since we then have the third equation at the bottom of the page, we only need to compare the elements at the last row of $\mathbf{EG}(z)$ and the elements at the last row of $\mathbf{G}(z)\mathbf{V}(z)$. It can be verified that $B_{N,m}(z)$, which is the mth element of the last row of the matrix $\mathbf{G}(z)\mathbf{V}(z)$, is given by

$$\begin{aligned} B_{N,m}(z) & \\ = (-1)^{N-1}e_{1,m+1}z^{-N} &+ \sum_{n=2}^{N-1}((-1)^{N-n+1}e_{n-1,m} \\ &+ (-1)^{N-n}e_{n,m+1})z^{-N+n-1} + e_{N-1,m}z^{-1}. \end{aligned}$$

From $\mathbf{EG}(z) = \mathbf{G}(z)\mathbf{V}(z)$, we have

$$B_{N,m}(z) = e_{N,m} + e_{N,m+1}z^{-1}.$$

This implies the following equations:

$$e_{1,m+1} = 0, \quad \text{and} \quad e_{N,m} = 0, \qquad \text{for } m = 1, 2, \ldots, N-1$$

and

$$e_{n,m} = e_{n+1,m+1} \qquad \text{for } m = 1, 2, \ldots, N-1, \quad n = 1, 2, \ldots, N-1.$$

It is not hard to see that these equations imply that $e_{m,n} = 0$ for $n \neq m$ and $e_{n,n} = e_{1,1}$ for all n. In other words, we have proved that $\mathbf{E} = \alpha\mathbf{I}$ for some constant α. By Theorem 2, $\mathbf{G}(z)$ is ambiguity resistant. ☐

Notice that in the above, the ambiguity-resistant precoder family has parameters $K = N-1$, which is the most interesting and important case of the precoders because the minimum bandwidth expansion in the precoding is usually desired.

$$\mathbf{G}_L(z) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -z^{-1} & 1 & 0 & \cdots & 0 & 0 & 0 \\ z^{-2} & -z^{-1} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ (-1)^{N-2}z^{-N+2} & (-1)^{N-3}z^{-N+3} & (-1)^{N-4}z^{-N+4} & \cdots & -z^{-1} & 1 & 0 \end{bmatrix}_{(N-1)\times N}.$$

$$\mathbf{EG}(z) = \begin{bmatrix} e_{1,1} + e_{1,2}z^{-1} & e_{1,2} + e_{1,3}z^{-1} & \cdots & e_{1,N-1} + e_{1,N}z^{-1} \\ e_{2,1} + e_{2,2}z^{-1} & e_{2,2} + e_{2,3}z^{-1} & \cdots & e_{2,N-1} + e_{2,N}z^{-1} \\ \vdots & \vdots & & \vdots \\ e_{N,1} + e_{N,2}z^{-1} & e_{N,2} + e_{N,3}z^{-1} & \cdots & e_{N,N-1} + e_{N,N}z^{-1} \end{bmatrix}.$$

$$\mathbf{G}(z)\mathbf{G}_L(z) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ (-1)^{N-1}z^{-N+1} & (-1)^{N-2}z^{-N+2} & \cdots & z^{-1} & 0 \end{bmatrix}$$

C. System Identifiability

With the establishment of ambiguity resistant precoders, we now give a set of sufficient conditions for blind identifiability of the system in Fig. 5.

Theorem 6: The system depicted in Fig. 5 is blindly identifiable when we have the folowing.

1) $\mathbf{G}(z)$ is ambiguity resistant.
2) $\mathbf{H}(z)$ is irreducible.
3) $\mathbf{G}(z)$ has order $Q \geq \lceil (N(R_o + Q_h)/K) - Q_h - R_o \rceil$, where $R_o = \lceil (NQ_h/M - N) \rceil$, and Q_h is the order of $\mathbf{H}(z)$.

Proof: If the ambiguous precoder output $\tilde{\mathbf{u}}(z)$ is determined, Condition 1 assures that $\mathbf{s}(z)$ can be identified within a scalar factor.

We will now prove that with Conditions 2 and 3, $\tilde{\mathbf{u}}(z)$ is blindly identifiable from the system outputs. To show this, it is important to realize that Theorem 1 is true only if the inputs to the channels $\mathbf{u}[n]$ have sufficient linear complexity. To be more specific, let us define the following convolutional matrices with R row blocks for $\mathbf{H}(z)$ and $\mathbf{G}(z)$, respectively

$$\mathcal{G}_R = \begin{bmatrix} \mathbf{G}[Q] & \cdots & \mathbf{G}[0] & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{G}[Q] & \cdots & \mathbf{G}[0] \end{bmatrix}$$
$$\mathcal{H}_R = \begin{bmatrix} \mathbf{H}[Q_h] & \cdots & \mathbf{H}[0] & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{H}[Q_h] & \cdots & \mathbf{H}[0] \end{bmatrix}. \qquad (10)$$

When $\mathbf{G}(z)$ and $\mathbf{H}(z)$ are irreducible, both $\mathcal{G}_R$ and $\mathcal{H}_R$ will eventually become full column rank as R increases.

The time domain representation of the undersampled system is given as follows:

$$\begin{aligned} &[\mathbf{y}^T[n] \quad \cdots \quad \mathbf{y}^T[n+R-1]]^T \\ &\quad = \mathcal{H}_R[\mathbf{u}^T[n-Q_h] \quad \cdots \quad \mathbf{u}^T[n+R-1]]^T \\ &\quad = \mathcal{H}_R\mathcal{G}_{R+Q_h}[\mathbf{s}^T[n-Q_h-Q] \quad \cdots \quad \mathbf{s}^T[n+R-1]]^T. \end{aligned} \qquad (11)$$

In order to determine $\{\mathbf{u}[n]\}$ within a matrix ambiguity, it is necessary for $\mathcal{H}_R$ to have full column rank while its corresponding input matrix

$$\begin{bmatrix} \mathbf{u}[n-Q_h] & \mathbf{u}[n-Q_h+1] & \cdots \\ \vdots & \vdots & \cdots \\ \mathbf{u}[n+R-1] & \mathbf{u}[n+R] & \cdots \end{bmatrix}$$

has full row rank. Assuming $\mathbf{s}[n]$ is i.i.d., this requires that $\mathcal{G}_{R+Q_h}$ has full row rank. Therefore, there must exist an R such that $\mathcal{H}_R$ has more rows than columns, whereas $\mathcal{G}_{R+Q_h}$ has more columns than rows. More specifically, the following two inequalities must hold simultaneously:

$$RM \geq (Q_h + R)N \qquad (12)$$
$$K(R + Q + Q_h) \geq N(R + Q_h) \qquad (13)$$

It can be easily verified that the above is true under Condition 3, which when combined with Condition 2, allows $\tilde{\mathbf{u}}(z)$ to be blindly identified. □

By now, one might want to ask whether there exists a family of precoders such that the undersampled system shown in Fig. 5 is blindly identifiable. The answer to this question is positive. When the order of a channel $\mathbf{H}(z)$ is known (the order of the actual channel could be lower), we are able to determine the order Q of a precoder based on Condition 3 in Theorem 6. Then, the ambiguity resistant precoder family proposed in Section III-B by replacing $\gamma = Q$ provides the precoders for the blind identifiability of the system in Fig. 5. This approach is actually used in Section V for numerical examples.

IV. ALGEBRAIC ALGORITHM DEVELOPMENT

Having discussed the identifiability of the undersampled multireceiver systems, we derive in this section an algebraic algorithm that can accomplish blind identification with a finite number of observations. Our objective herein is to demonstrate the feasibility of closed-form solutions of the problem under investigation. For this purpose, we only consider noise-free data without claiming anything concerning the optimality of the algorithm, although simulation results indicate that the proposed approach performs well even in the presence of noise.

Since the ambiguous precoder output $\{\tilde{\mathbf{u}}[n]\}$ can be identified using one of the existing multichannel blind identification algorithms, e.g., [5], [8], we limit ourselves to the problem of removing the matrix ambiguity from $\{\tilde{\mathbf{u}}[n]\}$.

Given a finite collection of the ambiguous precoder outputs $\{\tilde{\mathbf{u}}[n]\}_{n=0}^{R-1}$, it is not difficult to establish the following relations from (5):

$$\mathrm{diag}(\mathbf{T} \quad \cdots \quad \mathbf{T}) \begin{bmatrix} \tilde{\mathbf{u}}[0] \\ \vdots \\ \tilde{\mathbf{u}}[R-1] \end{bmatrix} = \begin{bmatrix} \mathbf{u}[0] \\ \vdots \\ \mathbf{u}[R-1] \end{bmatrix} = \mathcal{G}_R \underbrace{\begin{bmatrix} \mathbf{s}[-Q] \\ \vdots \\ \mathbf{s}[R-1] \end{bmatrix}}_{\underline{\mathbf{s}}}. \qquad (14)$$

Upon denoting $\mathbf{t}_i$ as the ith column of $\mathbf{T}$, $\underline{\mathbf{t}} = [\mathbf{t}_1^T \ \cdots \ \mathbf{t}_N^T]^T$, and $\tilde{\mathbf{U}}$ as the Kronecker product of $[\tilde{\mathbf{u}}[0] \ \cdots \ \tilde{\mathbf{u}}[R-1]]^T$ and $\mathbf{I}_{N \times N}$, we may rearrange (14) with respect to its unknowns, namely, $\underline{\mathbf{s}}$ and $\underline{\mathbf{t}}$, and obtain

$$[-\mathcal{G}_R \quad \tilde{\mathbf{U}}] \begin{bmatrix} \underline{\mathbf{s}} \\ \underline{\mathbf{t}} \end{bmatrix} = \mathbf{0}. \qquad (15)$$

Since we have NR equations with $(R+Q)K + N^2$ unknowns, the above equation set becomes *overdetermined* as R increases, provided that $N > K$. The system can be identified using simple least squares fitting when $\mathbf{G}(z)$ is ambiguity resistant.

Although (15) solves both $\underline{\mathbf{s}}$ and $\underline{\mathbf{t}}$, in many cases, $\mathbf{T}$ is the matrix of interest. To estimate $\mathbf{T}$ only, notice that $\underline{\mathbf{s}}$ can be expressed as

$$\underline{\mathbf{s}} = \mathcal{G}_R^{\dagger}\tilde{\mathbf{U}}\underline{\mathbf{t}}$$

TABLE I
CHANNEL COEFFICIENTS

	0	1	2	3	4	5
$h_0[n]$	.031 - .033i	.030 + .111i	.100 + .058i	.149 + .143i	-.032 + .042i	-.063 - .019i
$h_1[n]$	.041 - .019i	.332 - .106i	-.017 + .037i	-.085 + .055i	-.0048 - .0795i	.0081 - .3044i
$h_2[n]$	.003 + .045i	-.248 + .240i	-.156 - .091i	-.105 - .130i	.014 + .025i	.039 - .085i
$h_3[n]$	-.033 - .030i	-.144 - .122i	.209 - .098i	.099 - .012i	.0222 + .0143i	-.0563 + .1281i
$h_4[n]$	.025 - .038i	.404 - .224i	.259 + .033i	.293 + .029i	-.006 + .005i	-.064 - .081i
$h_5[n]$	.009 - .044i	-.005 - .247i	.004 - .075i	.040 + .001i	.009 + .073i	.113 - .073i
$h_6[n]$	-.044 + .008i	-.469 - .224i	.001 + .070i	-.098 + .074i	.040 - .009i	.118 + .136i
$h_7[n]$	.038 + .025i	.115 + .194i	.068 - .146i	.136 - .158i	.003 - .038i	-.027 - .024i

where † denotes the pseudo-inverse. Substituting the above into (15), $\underline{\mathbf{t}}$ can be determined as the least significant singular vector of

$$(\mathbf{I}-\mathcal{G}_R\mathcal{G}_R^{\dagger})\tilde{\mathbf{U}}. \tag{16}$$

By precalculating $(\mathbf{I}-\mathcal{G}_R\mathcal{G}_R^{\dagger})$, the computations in the estimation of $\mathbf{t}$ or $\mathbf{T}$ can be significantly reduced.

The above identification procedure can be summarized as follows.

1) Determine the precoder output vectors within an $N \times N$ matrix using any existing MIMO blind identification method (e.g., the subspace approach in [8] and [12]).
2) Form a linear equation set using the ambiguous precoder output vectors $\{\tilde{\mathbf{u}}[n]\}_{n=0}^{R-1}$ as in (15).
3) Determine elements of the ambiguity matrix from the the least significant singular vector of (16).
4) Recover the message signals as $\mathbf{s}(z)=\mathbf{G}_L(z)\mathbf{T}^{-1}\tilde{\mathbf{u}}(z)$, where $\mathbf{G}_L(z)$ is the left pseudo inverse of $\mathbf{G}(z)$ as appeared before.

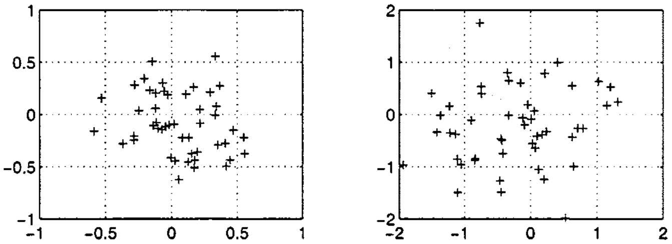
V. NUMERICAL EXAMPLES

Some numerical results are presented in this section to validate the identifiability and the efficacy of the proposed algorithms. All examples involved an eight-antenna system with the undersampling rate 3. The following ambiguity resistant precoder described in Section III-B was used:

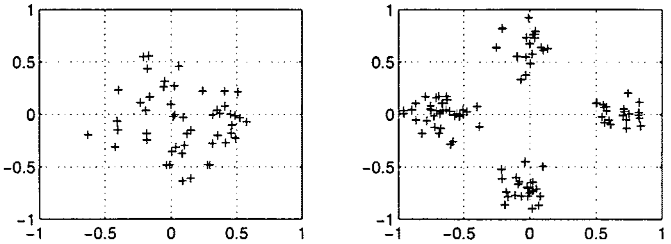
$$\mathbf{G}(z)=\begin{bmatrix}1 & 0\\ z^{-2} & 1\\ 0 & z^{-2}\end{bmatrix}.$$

The system simulated is thus $[(2,3);(3,8)]$, where the order of the channel $\mathbf{H}(z)$ is 2 for the eight antennas with the coefficients shown in Table I. The order of the above precoder $\mathbf{G}(z)$ is 2, which satisfies the sufficient condition 3 in Theorem 6. By the result in Section III-B, this precoder is also ambiguity resistant. In addition, it is not hard to check that the channel matrix $\mathbf{H}(z)$ is irreducible. Therefore, all sufficient conditions in Theorem 6 are satisfied, and thus, this undersampled system is blindly identifiable by Theorem 6.

The simulation results are the following. Table I summarizes the channel coefficients used in the simulations. The closed-form input estimation approach described in [8] was used to determine the ambiguous precoder output $\tilde{\mathbf{u}}[n]$. Only 30 estimated vectors $\tilde{\mathbf{u}}[n]$ were applied to the proposed method to determine the matrix ambiguity $\mathbf{T}$. The inputs were then recovered as



(a) Antenna 1 Output (b) Antenna 5 Output



(c) Recovered Precoder Output (d) Recovered Signal

Fig. 6. Signal constellations before and after blind recovery.

$$\hat{\mathbf{s}}(z)=\mathbf{G}_L(z)(\mathbf{T}^{-1}\tilde{\mathbf{u}}(z)).$$

Fig. 6 compares the signal constellations of the antenna outputs, the recovered precoder outputs, and the recovered signals. Even at 20 dB SNR, the antenna outputs are heavily distorted, due to the unknown channel effects and, more importantly, undersampling. As shown in Fig. 6(c), existing approaches can only restore the transmitted signals, i.e., the precoder outputs, within an matrix ambiguity. However, with precoding and the algorithm presented in this paper, the symbol sequence can be blindly recovered without significant increase in bandwidth.

Fig. 7 shows how the mean-square error (MSE) of the symbol estimates varies with the SNR. The results are obtained by averaging over 500 independent trials.

VI. CONCLUDING REMARKS

Blind channel identification has been traditionally premised on the output diversities of a communication system. When the system output is undersampled, only partial information of the

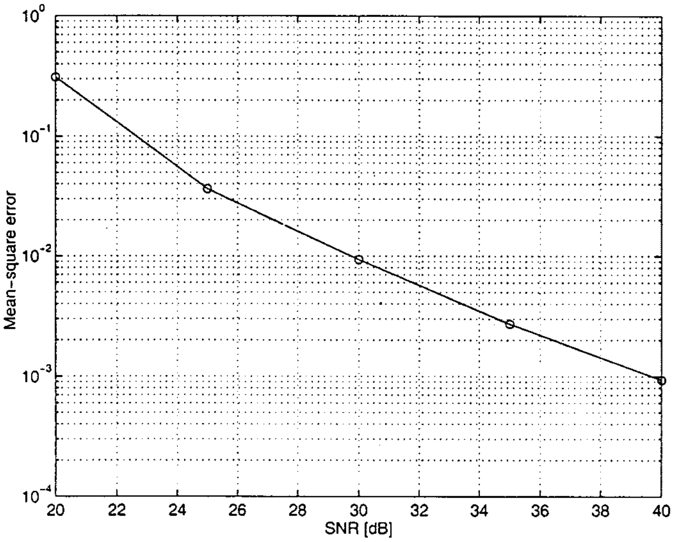


Fig. 7. MSE versus SNR.

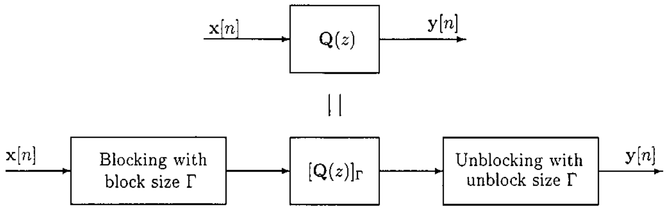


Fig. 8. Blocking an MIMO Ssystem.

input is contained in the observations, and this loss of information cannot be restored in general by using multiple receivers. In this paper, we have shown that by introducing redundancy, albeit minimum, at the input through precoding techniques, blind identification can be accomplished for undersampled systems in most scenarios. An important concept on precoders, i.e., *ambiguity resistant* precoders, has been introduced and used in the blind identification. Some necessary and sufficient conditions for ambiguity resistant precoders have been given, and in the meantime, a family of such precoders has been presented. An algebraic algorithm that determines the unknowns of an undersampled system with a finite number of data samples is also presented.

APPENDIX A
PROOF OF LEMMA 1

The lemma can be easily proved by using basic blocking operations on an MIMO LTI system.

Referring to Fig. 8, an MIMO system with transfer function $\mathbf{Q}(z)$ can be alternatively represented as a blocked system with transfer function $[\mathbf{Q}(z)]_\Gamma$ matrix, whose elements are the polyphase components of $\mathbf{Q}(z)$ similarly defined in (2) (see also [21]). $[\mathbf{Q}(z)]_\Gamma$ is termed the *blocked version* of the MIMO system with transfer function $\mathbf{Q}(z)$.

Let $\mathbf{Q}_\gamma(z)$ be the γth polyphase component matrix of $\mathbf{Q}(z)$, $\gamma = 0, 1, \ldots, \Gamma$, where $[\mathbf{Q}(z)]_\Gamma$ is given by the following block pseudo-circulant matrix:

$$[\mathbf{Q}(z)]_\Gamma = \begin{bmatrix} \mathbf{Q}_0(z) & z^{-1}\mathbf{Q}_{\Gamma-1}(z) & \cdots & z^{-1}\mathbf{Q}_1(z) \\ \mathbf{Q}_1(z) & \mathbf{Q}_0(z) & \cdots & z^{-1}\mathbf{Q}_2(z) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{Q}_{\Gamma-2}(z) & \mathbf{Q}_{\Gamma-3}(z) & \cdots & z^{-1}\mathbf{Q}_{\Gamma-1}(z) \\ \mathbf{Q}_{\Gamma-1}(z) & \mathbf{Q}_{\Gamma-2}(z) & \cdots & \mathbf{Q}_0(z) \end{bmatrix}^T. \tag{A.1}$$

With the above observation, we will prove Lemma 1 by showing the following. 1) Any $\{[(K,N);(L,M)] : N > K, MN > LK\}$ system can be transferred into a $\{[(\underline{K},\underline{N});(\underline{N},\underline{M})] : \underline{N} > \underline{K}, \underline{MN} > \underline{NK}\}$ system using blocking operations. 2) Any $[(K,N);(N,M)]$ system with $N > K$ and $MN > NK$ can be transferred into a $[(K,\underline{N});(\underline{N},\underline{M})]$ system with $K < \underline{N} < \underline{M}$.

Step 1) Referring to Fig. 4, if $N \neq L$, we may choose to block L precoder output vectors into a $NL \times 1$ vector, leading to an equivalent precoder filter, $[\mathbf{G}(z)]_L$. Its output vector can be viewed as blocked input vectors to $\mathbf{H}(z)$ with block size N. Therefore, the system is equivalent to a $[(KL,NL);(NL,NM)]$ system with the precoder $[\mathbf{G}(z)]_L$ and an unknown channel $[\mathbf{H}(z)]_N$.

Step 2) Now, consider a $[(K,N);(N,M)]$ system with $N > K$ and $MN > NK$. We may block the input vectors with blocking size $2N$, leading to a block version of the precoder $[\mathbf{G}(z)]_{2N}$ with dimension $2N^2 \times 2KN$. Accordingly, the channel matrix can be block with size $2N$, leading to a block version of the channel matrix $[\mathbf{H}(z)]_{2N}$ with size $2MN \times 2N^2$. Since $MN > NK$, there exist an integer R such that $2MN > R > 2KN$ (the factor 2 here guarantees that R is an integer). Note that the precoding matrix $[\mathbf{G}(z)]_{2N}$ can always be split into two parts—$[\mathbf{G}(z)]_{2N} = \mathbf{G}_1(z)\mathbf{G}_2(z)$—where $\mathbf{G}_1(z)$ has size $2N^2 \times R$, and $\mathbf{G}_2(z)$ has size $R \times 2KN$. Therefore, we may form a new precoder $\mathbf{G}(z) = \mathbf{G}_2(z)$ and a new channel matrix $\mathbf{H}(z) = [\mathbf{H}(z)]_{2N}\mathbf{G}_1(z)$ with more outputs than inputs. The new system is of $[(2KN,R);(R,2MN)]$ with $2KN < R < 2MN$.

APPENDIX B
PROOF OF THEOREM 3

We first prove the necessity of Condition 1. Assume there are a full-rank constant matrix $\mathbf{E}$ and an invertible $K \times K$ polynomial matrix $\mathbf{V}(z)$ such that the first column in the matrix $\mathbf{EG}(z)\mathbf{V}(z)$ is $(1,0,0,\ldots,0)^T$, i.e.,

$$\mathbf{EG}(z)\mathbf{V}(z) = \begin{bmatrix} 1 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}.$$

Then, by doing column elementary operations, the first row of the above matrix can be reduced to 0, except for the first element, i.e., there is a unimodular matrix $\mathbf{V}_1(z)$ such that

$$\mathbf{E}\mathbf{G}(z)\mathbf{V}(z)\mathbf{V}_1(z) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}.$$

Define an $N \times N$ nonidentical full-rank constant matrix $\mathbf{F}_l$ and a $K \times K$ nonidentical full-rank constant matrix $\mathbf{F}_r$ as follows:

$$\mathbf{F}_l = \text{diag}(2, 1, 1, \ldots, 1)_{N \times N}, \quad \text{and}$$
$$\mathbf{F}_r = \text{diag}(2, 1, 1, \ldots, 1)_{K \times K}.$$

Then, we have

$$\begin{aligned} \mathbf{F}_l\mathbf{E}\mathbf{G}(z)\mathbf{V}(z)\mathbf{V}_1(z)\mathbf{F}_r^{-1} &= \mathbf{F}_l \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix} \mathbf{F}_r^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix} \\ &= \mathbf{E}\mathbf{G}(z)\mathbf{V}(z)\mathbf{V}_1(z). \end{aligned}$$

Define

$$\tilde{\mathbf{E}} = \mathbf{E}^{-1}\mathbf{F}_l\mathbf{E}, \quad \text{and}$$
$$\mathbf{X}(z) = \mathbf{V}(z)\mathbf{V}_1(z)\mathbf{F}_r\mathbf{V}_1^{-1}(z)\mathbf{V}^{-1}(z).$$

Clearly, $\tilde{\mathbf{E}} \neq \alpha\mathbf{I}$ and $\mathbf{X}(z) \neq \beta\mathbf{I}$ for any constants α and β because $\mathbf{F}_l \neq \gamma\mathbf{I}$ and $\mathbf{F}_r \neq \eta\mathbf{I}$ for any constants γ and η when $K > 1$. However, we have $\tilde{\mathbf{E}}\mathbf{G}(z) = \mathbf{G}(z)\mathbf{X}(z)$, which contradicts with the ambiguity resistancy of $\mathbf{G}(z)$. This proves the necessity of Condition 1.

Condition 2 can be proved by the following argument. When $K = N$, decompose $\mathbf{G}_{K \times K}$ into $\mathbf{G} = \mathbf{W}_{K \times K}\mathbf{V}_{K \times K}$ using the Smith–McMillan decomposition. Since $\mathbf{G}(z)$ is unimodular, its Smith–McMillan decomposition has an identity matrix as the middle factor [21]. It can be easily verified that for any full-rank matrix $\mathbf{E} \neq \mathbf{I}$

$$\mathbf{G} = \underbrace{\mathbf{E}\mathbf{W}}_{\mathbf{W}_2}\underbrace{\mathbf{W}^{-1}\mathbf{E}^{-1}\mathbf{W}\mathbf{V}}_{\mathbf{V}_2}$$

is another valid Smith–McMillan decomposition. Clearly, $\mathbf{E}\mathbf{W} \neq \mathbf{W}$ (the precoder) is thus not ambiguity resistant, according to Theorem 4, which is presented below. A similar argument applies to $K > N$.

We now prove Condition 3. Let $\mathbf{G}(z) = \sum_{n=0}^{Q} \mathbf{G}[n]z^{-i}$, and consider a special case when the $\mathbf{X}(z)$ in (7) is a constant: $\mathbf{X}(z) = \mathbf{X}$. Then, (7) can be rewritten as

$$\mathbf{E}\mathbf{G}(z) = \mathbf{G}(z)\mathbf{X}$$

or

$$\begin{aligned} &\mathbf{E}\mathbf{G}[0] + \mathbf{E}\mathbf{G}[1]z^{-1} + \cdots + \mathbf{E}\mathbf{G}[Q]z^{-Q} \\ &\quad = \mathbf{G}[0]\mathbf{X} + \mathbf{G}[1]\mathbf{X}z^{-1} + \cdots + \mathbf{G}[Q]\mathbf{X}z^{-Q}. \end{aligned}$$

For the above equation to hold, we must have

$$\begin{aligned} &\mathbf{E}[\mathbf{G}[0] \quad \mathbf{G}[1] \quad \cdots \quad \mathbf{G}[Q]] \\ &\quad = [\mathbf{G}[0] \quad \mathbf{G}[1] \quad \cdots \quad \mathbf{G}[Q]] \begin{bmatrix} \mathbf{X} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{X} \end{bmatrix}. \end{aligned} \tag{B.2}$$

Letting the parameters in $\mathbf{E}$ and $\mathbf{X}$ be the unknowns, $\mathbf{G}(z)$ is not ambiguity resistant if the above equation set has nonunique solutions. Equation (B.2) only defines $N \times K(Q+1)$ linear equations, whereas there are $N^2 + K^2$ unknowns. Accounting for the one degree of freedom due to the scalar ambiguity, a necessary condition for a unique solution of (B.2) is that $NK(Q+1) \geq N^2 + K^2 - 1$. Hence, $Q \geq (N^2+K^2-1/NK)-1$. □

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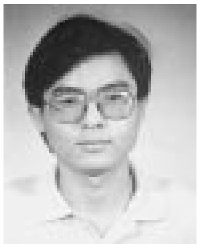

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