Uplink Channel Capacity of Space-Division-Multiple-Access Schemes

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***Abstract*— Signal-to-noise ratio (SNR) and signal bandwidth have been viewed as the dominant factors determining channel capacity. In wireless communications, the channel capacity can be increased for a given SNR and a given spectral region, by exploiting the spatial diversity provided by the use of multiple antennas and transceivers at a base station. In this paper, we calculate the channel capacity enhancement of a so-called Space-Division-Multiple-Access (SDMA) system and investigate its dependence with respect to different decoding schemes, terminal positions, and receiver numbers. Inner and outer capacity boundaries for joint decoding and independent decoding are presented, along with physical explanations as to how these boundaries can be achieved. We show that exploitation of the spatial diversity not only increases the overall achievable rates of both joint and independent decoding, but also closes the gap between their corresponding capacity regions, thus bringing the performance of the low-cost independent decoding scheme close to that of the *optimal* joint decoding. Practical issues of optimum projection and power control are also briefly addressed.**

***Index Terms*— Antenna array, channel capacity, multiple-access channel, power control, spatial diversity multiple access, wireless communications.**

I. Introduction

A prime issue in current wireless systems is the conflict between the finite spectrum available and the increasing demands for wireless services. To mitigate this problem, various multiplexing schemes (e.g., time-division multiple access (TDMA) and code-division multiple access (CDMA)) have been proposed to increase the channel capacity for a given amount of spectrum. Most research efforts are focused on searching for more efficient ways of using the existing time–frequency resources [1]–[8]. Recently, it has been proposed that the use of multiple receivers (or of an antenna array) at the base station can significantly increase the channel capacity by exploiting the spatial diversity among different users [5], [9]–[11]. Such a system has been referred to as Spatial-Diversity Multiple Access (or SDMA). Significant progress has been made in incorporating antenna arrays into existing wireless communication systems. However, one fundamental question remains unanswered: "What is (an upper bound on) the channel capacity of an SDMA system?" or "How tightly can we pack information in space?".

In this paper, we present a study on the channel capacity of an SDMA system from an information-theoretic viewpoint. In particular, we derive the *capacity region*, i.e., the closure of the class of achievable rate vectors [7], of an antenna array multiple-access system. Since the capacity region defines the limit of error-free communications given certain channel characteristics, it is used as the ultimate measure of channel capacity in this field. Our objective is to evaluate the increase in channel capacity made possible by using multiple receivers, and further to examine the system performance under different wireless scenarios. The formulas we derived are based on an idealized one-cell model, which may seem restrictive, especially given that interference from neighboring cells is critical to cellular network operations. However, our motivation was that simple models can disclose the fundamentals of an SDMA system, and provide insights into system behavior.

The paper is organized as follows. Section II presents a mathematical model for and some basic assumptions on a multireceiver multiple-access Channel (MAC) that will be used throughout this paper. Section III derives the capacity regions for an SDMA system. A heuristic study on the channel capacities of a simple two-user system is provided in Section IV. The analysis is then extended to a general P-user system in Section V. In Section VI, several practical issues such as optimal projection (combining) and capacity sensitivity to power variations are discussed.

II. A Mathematical Model

We consider a one-cell network where mobiles (users) transmit to the base station. We look at the mobile-to-base-station link (uplink) and model it as a multiple-access system with Gaussian channels. Further, multiple receivers are used at the base station to exploit the spatial diversity among multiple terminals.

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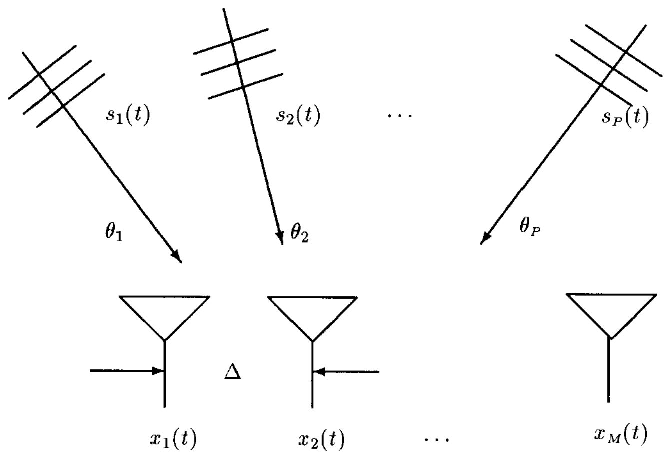

Fig. 1. A typical array scene.

A. The Antenna Array

Without loss of essential generality, we consider only a uniform linear array, i.e., M sensors uniformly spaced with displacement Δ. Fig. 1 shows a typical MAC with an array of receivers. For tractability in the subsequent analysis, we invoke the following usual assumptions concerning the signal and noise, as well as the system configuration [6].

- The sources are sufficiently far from the array (far-field sources) that the wavefronts are planar. Also, the incoming signals are narrowband,[1] i.e., the baseband signal does not vary over the length of the array except for a phase shift.
- The noise components observed at all the receivers have identical, independent Gaussian distributions and are spatially and temporally white.
- The signals from different users are independent and Gaussianly distributed and are temporally white.
- The array response to each source is slowly varying and hence is constant over many information bits.

In the absence of noise, the array output vector corresponding to a single source $s_1(t)$ can be written as

$$\boldsymbol{x}(t) \stackrel{\text{def}}{=} [x_1(t) \quad x_2(t) \quad \cdots \quad x_M(t)]^T = \boldsymbol{a}_1 s_1(t),$$

where $(\cdot)^T$ denotes transposition, and

$$\boldsymbol{a}_1 = [\boldsymbol{a}_1(1)\ \boldsymbol{a}_1(2) \cdots \boldsymbol{a}_1(M)]^T$$

is the array response vector.

When there is only a direct-path component associated with $s_1(t)$, two neighboring elements in $\boldsymbol{a}_i$ differ only by a constant phase shift $e^{2\pi\theta\Delta/\lambda}$ using complex representation, where λ is the carrier wavelength and θ is the direction-of-arrival (DOA) of the incoming signal. Normalizing each element in $\boldsymbol{a}_1$ with respect to the first element $\boldsymbol{a}_1(1)$ yields

$$\boldsymbol{a}_1 = \boldsymbol{a}_1(1) \underbrace{[1, e^{j2\pi \sin\theta\Delta/\lambda}, \cdots, e^{j(M-1)2\pi\sin\theta\Delta/\lambda}]^T}_{\boldsymbol{a}^T(\theta)}. \quad (1)$$

[1]This narrowband assumption is reasonable in cellular telephony since the carry frequency is usually 800 MHz while the bandwidth of the information transmission from each terminal is usually less than 2 MHz.

The above $\boldsymbol{a}(\theta)$ is termed as the *steering vector*; it is a function of the DOA and the array configuration. In a direct-path-only environment, the array response vector is identical to the steering vector (up to a complex scalar). In most practical situations, however, $\boldsymbol{a}_1$ contains both the direct-path and multipath components

$$\boldsymbol{a}_1 = \boldsymbol{a}(\theta_{1,1}) + \sum_{l=2}^{L_1} \alpha_l \boldsymbol{a}(\theta_{1,l}) \quad (2)$$

where L_1 denotes the total number of paths associated with $s_1(t)$, α_l represents the amplitude and phase difference between the lth multipath component and the direct path at $\theta_{1,1}$. To facilitate our derivations in the remainder of this paper, we normalize $\boldsymbol{a}_1$ so that $\|\boldsymbol{a}_1\| = \sqrt{M}$. For a multiuser system with additive noise, $\boldsymbol{x}(t)$ can be written as

$$\boldsymbol{x}(t) = \sum_{k=1}^{P} \boldsymbol{a}_k s_k(t) + \boldsymbol{n}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t) \quad (3)$$

$$\begin{aligned} \boldsymbol{A} &= [\boldsymbol{a}_1, \cdots, \boldsymbol{a}_P] \\ \boldsymbol{s}(t) &= [s_1(t), \cdots, s_P(t)]^T \\ \boldsymbol{n}(t) &= [n_1(t), n_2(t), \cdots, n_M(t)]^T \end{aligned} \quad (4)$$

where P is the number of users, $\boldsymbol{n}(t)$ is the noise vector, and $\boldsymbol{A}$ is defined as the *array manifold* whose columns are the array response vectors. For simplicity, we normalize the noise power to unity and define

$$\begin{aligned} \boldsymbol{R}_S &= E\{\boldsymbol{s}(t)\boldsymbol{s}^H(t)\} = \text{diag}\{\sigma_1^2, \cdots, \sigma_P^2\} \\ \boldsymbol{R}_N &= E\{\boldsymbol{n}(t)\boldsymbol{n}^H(t)\} = \boldsymbol{I} \end{aligned} \quad (5)$$

where $(\cdot)^H$ denotes Hermitian. Consequently,

$$\boldsymbol{R}_x = E\{\boldsymbol{x}(t)\boldsymbol{x}^H(t)\} = \boldsymbol{A}\boldsymbol{R}_S\boldsymbol{A}^H + \boldsymbol{I}. \quad (6)$$

III. CAPACITY REGIONS OF SDMA SYSTEMS

The problem under consideration is at what rates we can *reliably* transmit P signals to an array of multiple receivers in a standard Gaussian channel with a given signal-to-noise ratio (SNR). Of course, the system capacity also depends on how the decoding is performed. In this paper, we only discuss two types of decoding methods: *joint decoding* and *independent decoding*. Joint decoding means that decoding of all signals is performed simultaneously, while independent decoding means that different signals are decoded independently and in parallel. Though joint decoding is optimal, it requires $O(2^P)$ computational complexity which may be too expensive to implement. Independent decoding, on the other hand, requires $O(P)$ complexity and thus may be more practical. The fundamental difference between joint and independent decoding is that the former regards all $\{s_k(t)\}$ as signals whereas the latter considers $\{s_k(t)\}$, $k \neq i$, as interference when decoding $s_i(t)$.

A. Capacity Region for Joint Decoding

Let T denote a subset of $(1, \cdots, P)$ and T^c its complement, i.e., $T \cap T^c = \emptyset$, $T \cup T^c = (1, \cdots, P)$. We define

- $\boldsymbol{A}_T$ (respectively, $\boldsymbol{A}_{T^c}$) to be the matrix whose columns are array response vectors of the sources in T (T^c).

- $\boldsymbol{R}_{S,T}$ (respectively, $\boldsymbol{R}_{S,T^c}$) to be the covariance matrix of the sources in T (T^c).

The covariance matrix in (6) can be decomposed as

$$\boldsymbol{R}_x = \boldsymbol{I} + \boldsymbol{A}_T \boldsymbol{R}_{S,T} \boldsymbol{A}_T^H + \boldsymbol{A}_{T^c} \boldsymbol{R}_{S,T^c} \boldsymbol{A}_{T^c}^H.$$

Using the above notation and recognizing that the system depicted in Fig. 1 is a multiple-access channel (MAC) with parallel channels, any achievable rate P-tuple $(R_1, \cdots, R_P)$ corresponding to the P users must satisfy

$$\begin{aligned}\sum_{k\in T} R_k &\le \max I(\boldsymbol{x}(t); s_k(t), k \in T | s_l(t), l \in T^c) \\ &\le \tfrac{1}{2}\log(\det[\boldsymbol{I} + \boldsymbol{A}_T \boldsymbol{R}_T \boldsymbol{A}_T^H]) \end{aligned} \tag{7}$$

where $I(x;y|z)$ denotes the mutual information between x and y, given z.

Proof: The first part of the inequality comes from standard information theory (see, e.g., [7], [12]).

For the second part, we have

- $I(\boldsymbol{x}(t); s_k(t), k \in T | s_l(t), l \in T^c)$
  $= H(\boldsymbol{x}(t)|s_l(t), l \in T^c) - H(\boldsymbol{x}(t)|s_k(t), k \in [1 \cdots P]);$
- $H(\boldsymbol{x}(t)|s_k(t), k \in [1 \cdots P]) = H(\boldsymbol{n}(t)) = \dfrac{1}{2}\log(2\pi e)^P$
  by the assumption of Gaussian noise with $\boldsymbol{R}_N = \boldsymbol{I}$;
- $H((\boldsymbol{x}(t)|s_l(t), l \in T^c)$
  $= \dfrac{1}{2}\log(2\pi e)^P \det[\boldsymbol{I} + \boldsymbol{A}_T \boldsymbol{R}_{S,T} \boldsymbol{A}_T^H])$
  by the assumption of Gaussian signals and noise. □

Here $H(\boldsymbol{x})$ and $H(\boldsymbol{x}|\boldsymbol{y})$ denote the entropy of a random vector $\boldsymbol{x}$, and the joint entropy of two random vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively [7].

The convex hull of the region delimited by (7) is the capacity region. The above results can be equivalently expressed as

$$\begin{aligned} R_k &\le \frac{1}{2}\log(\det(\boldsymbol{I} + \sigma_k^2 \boldsymbol{a}_k \boldsymbol{a}_k^H)) \\ R_{k_1} + R_{k_2} &\le \frac{1}{2}\log(\det(\boldsymbol{I} + \sigma_{k_1}^2 \boldsymbol{a}_{k_1} \boldsymbol{a}_{k_1}^H + \sigma_{k_2}^2 \boldsymbol{a}_{k_2} \boldsymbol{a}_{k_2}^H)) \\ &\vdots \qquad \vdots \\ \sum_{k=1}^{P} R_k &\le \frac{1}{2}\log\det(\boldsymbol{R}_x). \end{aligned} \tag{8}$$

B. Capacity Region for Independent Decoding

In the case of independent decoding, the MAC is an interfering channel. The decoder for $s_k(t)$ considers background noise as well as other signal sources as interference. The achievable rate satisfies

$$R_k \le \max I(\boldsymbol{x}(t); s_k(t)) = \frac{1}{2}\log\left(\frac{\det(\boldsymbol{R}_x)}{\det(\boldsymbol{R}_x - \sigma_k^2 \boldsymbol{a}_k \boldsymbol{a}_k^H)}\right). \tag{9}$$

Proof: Let $T = k$ and $T^c = [1, \cdots, k-1, k+1, \cdots, P]$.

- $\max I(\boldsymbol{x}(t); s_k(t)) = H(\boldsymbol{x}) - H(\boldsymbol{x}|s_k(t));$
- $H(\boldsymbol{x}) = \frac{1}{2}\log(2\pi e)^P \det(\boldsymbol{R}_x);$
- $H(\boldsymbol{x}|s_k(t)) = \frac{1}{2}\log(2\pi e)^P \det[\boldsymbol{I} + \boldsymbol{A}_{T^c} \boldsymbol{R}_{S,T^c} \boldsymbol{A}_{T^c}^H]$
  $= \frac{1}{2}\log(2\pi e)^P \det[\boldsymbol{R}_x - \sigma_k^2 \boldsymbol{a}_k \boldsymbol{a}_k^H].$

IV. Two-User Systems

With the above formulas, we can now investigate the performance of an SDMA system under different scenarios. But first, let us build up some intuition by examining a simple two-user system. Results from this section will be extended to the general case in the next section, where we examine SDMA system behavior and highlight some of its salient features.

A. Capacity Region

From (8) and (9) we see that the capacity regions are clearly functions of $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and of the users' SNR's. Since $\boldsymbol{I} + \boldsymbol{A}\boldsymbol{R}_s\boldsymbol{A}^*$, $\boldsymbol{I} + \sigma_1^2 \boldsymbol{a}_1 \boldsymbol{a}_1^*$, and $\boldsymbol{I} + \sigma_2^2 \boldsymbol{a}_2 \boldsymbol{a}_2^*$ are $M \times M$ matrices, their determinants can be difficult to evaluate. We can alleviate this difficulty using the following well-known lemmas.

Lemma 1: For any matrices $\boldsymbol{D}_1$ $(m \times n)$ and $\boldsymbol{D}_2$ $(n \times m)$

$$\det(\boldsymbol{I}f + \boldsymbol{D}_1 \boldsymbol{D}_2) = \det(\boldsymbol{I} + \boldsymbol{D}_2 \boldsymbol{D}_1). \tag{10}$$

Lemma 2: Let $\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$ and $\boldsymbol{A}$ be invertible, then

$$\det(\boldsymbol{M}) = \det(\boldsymbol{A}) \det(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}). \tag{11}$$

Joint Decoding: By Lemma 1

$$\begin{aligned} \det(\boldsymbol{I} + \sigma_k^2 \boldsymbol{a}_k \boldsymbol{a}_k^H) &= \det(1 + \sigma_k^2 \boldsymbol{a}_k^H \boldsymbol{a}_k) \\ &= (1 + M\sigma_k^2), \qquad k = 1, 2. \end{aligned} \tag{12}$$

$$\begin{aligned} \det(\boldsymbol{I} + \boldsymbol{A}\boldsymbol{R}_S \boldsymbol{A}^H) &= \det(\boldsymbol{I} + \boldsymbol{R}_S \boldsymbol{A}^H \boldsymbol{A}) \\ &= \det\begin{bmatrix} 1 + M\sigma_1^2 & \sigma_1^2 \boldsymbol{a}_1^H \boldsymbol{a}_2 \\ \sigma_2^2 \boldsymbol{a}_2^H \boldsymbol{a}_1 & 1 + M\sigma_2^2 \end{bmatrix}. \end{aligned} \tag{13}$$

Scalar expressions for achievable rates of a two-user system follows immediately

$$\begin{aligned} R_1 &\le \frac{1}{2}\log(1 + M\sigma_1^2); \\ R_2 &\le \frac{1}{2}\log(1 + M\sigma_2^2); \\ R_1 + R_2 &\le \frac{1}{2}\log(1 + M(\sigma_1^2 + \sigma_2^2) \\ &\quad + M^2 \sigma_1^2 \sigma_2^2 \sin^2 \angle(\boldsymbol{a}_1, \boldsymbol{a}_2)) \end{aligned} \tag{14}$$

where $\angle(\boldsymbol{a}_1, \boldsymbol{a}_2)$ denotes the angle between the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, and

$$\sin^2 \angle(\boldsymbol{a}_1, \boldsymbol{a}_2) = 1 - \frac{|\boldsymbol{a}_1^H \boldsymbol{a}_2|^2}{\|\boldsymbol{a}_1\|^2 \cdot \|\boldsymbol{a}_2\|^2}. \tag{15}$$

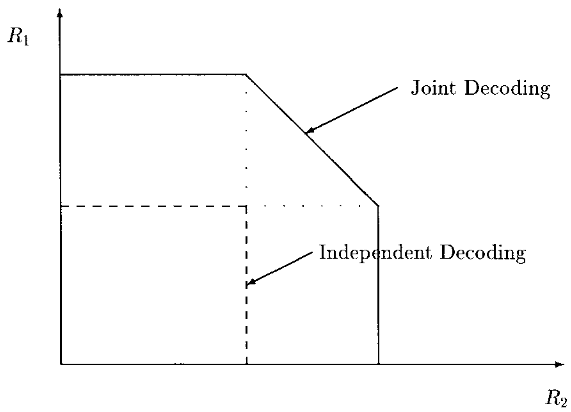

Fig. 2. Capacity regions for a two-user M-receiver system.

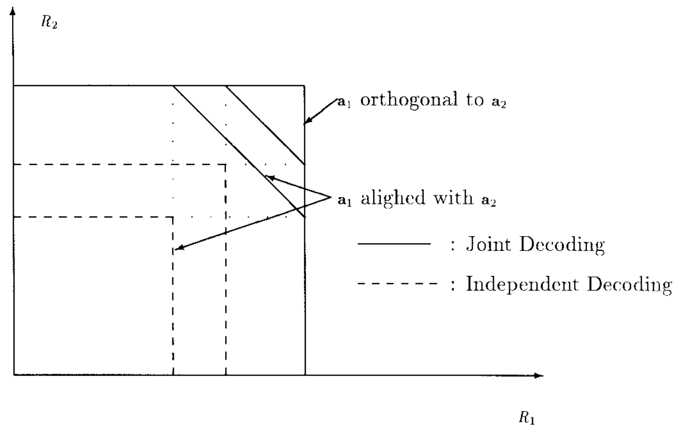

Fig. 3. Capacity regions with different users' positions.

Independent Decoding: Similarly, from (9) and with the help of Lemma 1, one can show that the capacity region for independent decoding is a rectangle bounded by

$$R_1 \le \frac{1}{2}\log\left(1+\frac{M\sigma_1^2+M^2\sigma_1^2\sigma_2^2\sin^2\angle(\boldsymbol{a}_1,\boldsymbol{a}_2)}{1+M\sigma_2^2}\right) \quad (16)$$

$$R_2 \le \frac{1}{2}\log\left(1+\frac{M\sigma_2^2+M^2\sigma_1^2\sigma_2^2\sin^2\angle(\boldsymbol{a}_1,\boldsymbol{a}_2)}{1+M\sigma_1^2}\right). \quad (17)$$

Typical capacity regions are depicted in Fig. 2. Clearly, both the regions will depend upon the physical locations, or more precisely the array response vectors, of the two users. In the following, we show that the capacity regions expand and vary between two boundaries as the positions of the users change.

B. Effect of Users' Positions

Outer Boundary: When the users' positions are such that the array response vectors are orthogonal to one another, $\sin^2\angle(\boldsymbol{a}f_1,\boldsymbol{a}f_2)=1$ and the capacity region for joint decoding is given by

$$R_1 \le \frac{1}{2}\log(1+M\sigma_1^2)$$
$$R_2 \le \frac{1}{2}\log(1+M\sigma_2^2) \quad (18)$$
$$R_1+R_2 \le \frac{1}{2}\log(1+M\sigma_1^2)+\frac{1}{2}\log(1+M\sigma_2^2). \quad (19)$$

For independent decoding

$$R_1 \le \frac{1}{2}\log(1+M\sigma_1^2) \qquad R_2 \le \frac{1}{2}\log(1+M\sigma_2^2). \quad (20)$$

These capacity regions are obviously maximal. Remarkably, the capacity region for independent decoding coincides with that for joint decoding, which has a well-known rectangular shape [7]. The MAC is orthogonal, which means that by using SDMA one can separate one signal from the other exactly.

Inner Boundary: When the users' positions are such that the array response vectors are aligned with each other, $\sin^2\angle(\boldsymbol{a}f_1,\boldsymbol{a}f_2)=0$ and the capacity region of joint decoding is given by

$$R_1 \le \frac{1}{2}\log(1+M\sigma_1^2)$$
$$R_2 \le \frac{1}{2}\log(1+M\sigma_2^2)$$
$$R_1+R_2 \le \frac{1}{2}\log(1+M(\sigma_1^2+\sigma_2^2)). \quad (21)$$

For independent decoding

$$R_1 \le I(s_1(t);\boldsymbol{x}f(t)) = \frac{1}{2}\log\left(1+\frac{M\sigma_1^2}{1+M\sigma_2^2}\right)$$
$$R_2 \le I(s_2(t);\boldsymbol{x}f(t)) = \frac{1}{2}\log\left(1+\frac{M\sigma_2^2}{1+M\sigma_1^2}\right). \quad (22)$$

These capacity regions are obviously the smallest over all possible $(\boldsymbol{a}_1,\boldsymbol{a}_2)$. However, one does gain a factor of M in SNR in joint decoding by using multiple antennas. This is because with M sensors, the receiver gets M replicas of the signal, which can be added up coherently whereas noise adds up incoherently. On the other hand, there is not much gain in the capacity for independent decoding, especially when σ_1^2 and σ_2^2 are low. The variation of capacity regions with respect to different users' positions are illustrated in Fig. 3.

V. P-User Systems

In this section, some important features of a P-user MAC are elucidated by theoretical derivations and are also highlighted by numerical examples.

A. Effect of Users' Positions

The Outer Boundary: From the discussions in the previous section, one may expect that the capacity region to be maximum when the array response vectors are orthogonal to each other. This expectation does hold, but only for certain limited cases. In particular, it is only true if the number of users is smaller than the number of receivers. In the following, we shall use the *symmetric* channel capacity of joint decoding, i.e., $\sum_{k=1}^{P} R_k$, to quantify the variation of the capacity regions.

Theorem 1: When the number of users P is less than the number of sensors M, the symmetric capacity region is maximum when the users' array response vectors are orthogonal to each other.

Proof: From (8), we need to prove that for a given $\boldsymbol{R}_s$, $\det(\boldsymbol{I}+\boldsymbol{A}\boldsymbol{R}_s\boldsymbol{A}^H)$ is maximized when the columns of $\boldsymbol{A}$ are orthogonal. We shall prove the theorem by induction.

- The argument is obviously true for the two-user system.
- Assume it is also true for $P-1$ users.
- Let us use subscripts to denote the number of users involved in various parameter quantity. By Lemmas 1 and 2, $\boldsymbol{R}_x$ in (8) becomes (see (23) at the bottom of this page), where

$$\boldsymbol{c} \stackrel{\text{def}}{=} (\sigma_1\sigma_P\boldsymbol{a}_1^H\boldsymbol{a}_P, \cdots, \sigma_{P-1}\sigma_P\boldsymbol{a}_{P-1}^H\boldsymbol{a}_P)^H.$$

The first term is proportional to the channel capacity for $(P-1)$ users and is thus maximized when the columns of $\boldsymbol{A}_{P-1}$ are orthogonal to each other. The second is maximized when $\boldsymbol{c}=\boldsymbol{0}$ since

$$(\boldsymbol{I}+\boldsymbol{R}_{s,P-1}^{1/2}\boldsymbol{A}_{P-1}^H\boldsymbol{A}_{P-1}\boldsymbol{R}_{s,P-1}^{1/2})^{-1}$$

is a positive-definite matrix. The condition $\boldsymbol{c}=\boldsymbol{0}$ means that all the columns of $\boldsymbol{A}_P$ are orthogonal to each other.

It is straightforward to show that the capacity region in this case is a cube in the P-dimensional space and is the same for both joint decoding and independent decoding.

For a general MAC, the number of users in the system may exceed the number of receivers. For example, in a CDMA system, the number of cochannel users can be increased by using longer spreading codes, whereas the employment of a large number of physical receivers may be prohibited. In such cases, the array response matrix $\boldsymbol{A}$ has more columns than rows. Consequently, it is no longer possible for all the array response vectors to be orthogonal to one another. The optimum users' array response vectors assumed in Theorem 1 are clearly invalid.

Theorem 2: When the number of users P is strictly larger than the number of sensors M, the symmetric channel capacity is maximum when the rows of the scaled array response matrix

$$\boldsymbol{A}=[\sigma_1\boldsymbol{a}_1, \sigma_2\boldsymbol{a}_2, \cdots, \sigma_P\boldsymbol{a}_P], \qquad k=1,\cdots,M$$

are orthogonal to one another.

Proof: Apply Lemma 1 to the Theorem 1 and the proof is straightforward. □

The Inner Boundary:

Theorem 3: The capacity region is minimum when the users' array response vectors are aligned with one another.

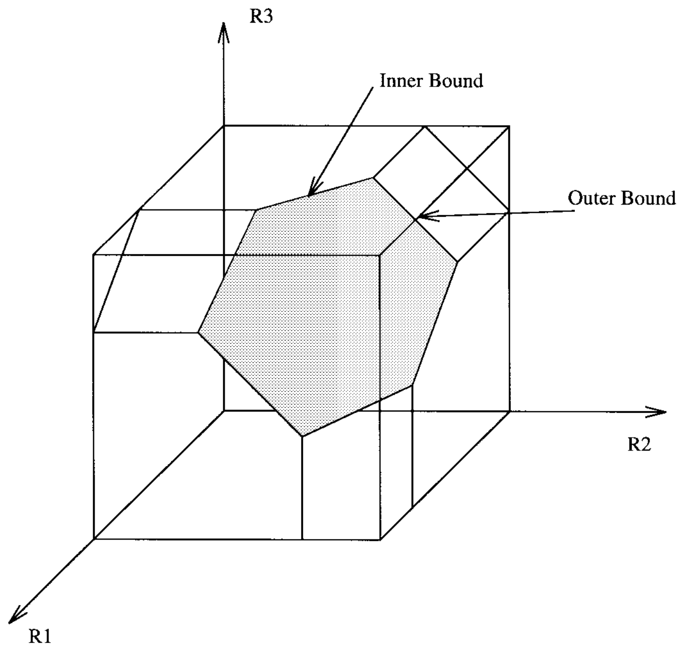

Fig. 4. Capacity regions for a three-user M-receiver ($M>3$) system.

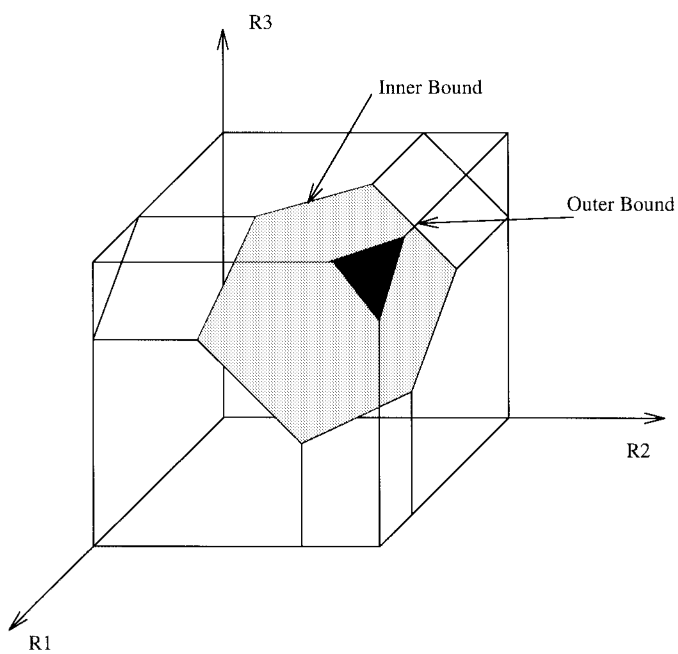

Fig. 5. Capacity regions for a three-user two-receiver system.

The proof of this theorem is very similar to that of Theorem 2 and will not be presented.

Typical capacity regions for a three-user system with $M>3$ and $M=2$ receivers are shown in Figs. 4 and 5, respectively.

$$\begin{aligned}\det(\boldsymbol{I}+\boldsymbol{A}_P\boldsymbol{R}_{s,P}\boldsymbol{A}_P^H) &= \det(\boldsymbol{I}+\boldsymbol{R}_{s,P}^{1/2}\boldsymbol{A}_P^H\boldsymbol{A}_P\boldsymbol{R}_{s,P}^{1/2})\\ &= \det\begin{pmatrix}(\boldsymbol{I}+\boldsymbol{R}_{s,P-1}^{1/2}\boldsymbol{A}_{P-1}^H\boldsymbol{A}_{P-1}\boldsymbol{R}_{s,P-1}^{1/2} & \boldsymbol{c}\\ \boldsymbol{c}^H & 1+M\sigma_P^2\end{pmatrix}\\ &= \det(\boldsymbol{I}+\boldsymbol{A}_{P-1}\boldsymbol{R}_{s,P-1}\boldsymbol{A}_{P-1}^H)(1+M\sigma_P^2-\boldsymbol{c}^H(\boldsymbol{I}+\boldsymbol{R}_{s,P-1}^{1/2}\boldsymbol{A}_{P-1}^H\boldsymbol{A}_{P-1}\boldsymbol{R}_{s,P-1}^{1/2})^{-1}\boldsymbol{c})\end{aligned} \qquad (23)$$

B. Effect of Number of Receivers

If more receivers are added, the additional signals received at the base station may lead to an expansion of the capacity regions. At the same time, the richer spatial diversity among the users may be captured by the enhanced array of receivers, which can further separate the originally close array response vectors, allowing the performance of independent decoding to approach to that of joint decoding. Here, we demonstrate that the capacity region for independent decoding will eventually merge with that for joint decoding as the number of receivers increases. Since independent decoding is more practical, such a result is important in real applications.

Theorem 4: When the number of receivers approaches infinity, the capacity region of independent decoding becomes identical to that of joint decoding with probability 1.

Proof: By Theorem 1, it is sufficient to show that the columns of $\lim_{M\to\infty} \boldsymbol{A}_M$ being orthogonal is an event whose probability is 1.

Denote $\phi = 2\pi \sin\theta\Delta/\lambda$ and rewrite the steering vector in (1) with M receivers more concisely as

$$\boldsymbol{a}_M(\phi) = [1, e^{j\phi}, \cdots, e^{j(M-1)\phi}]^T.$$

For any $\phi_i \neq \phi_j$

$$\begin{aligned}\sin^2 \angle(\boldsymbol{a}_M(\phi_i), \boldsymbol{a}_M(\phi_j)) &= 1 - \frac{|\boldsymbol{a}_M^H(\phi_i)\boldsymbol{a}_M(\phi_j)|^2}{|\boldsymbol{a}_M(\phi_i)|^2 \cdot |\boldsymbol{a}_M(\phi_j)|^2} \\ &= 1 - \frac{1}{M^2}\left|\frac{1-e^{jM(\phi_j-\phi_i)}}{1-e^{j(\phi_j-\phi_i)}}\right|^2.\end{aligned}$$

Clearly,

$$\lim_{M\to\infty} \sin^2 \angle(\boldsymbol{a}_M(\phi_i), \boldsymbol{a}_M(\phi_j)) = 1$$

which means that the steering vectors will eventually become orthogonal in a high-dimensional space. Since array response vectors are linear combinations of steering vectors, and the probability of identical DOA's from different users is 0, the array response vectors will be orthogonal to one another with probability 1 as $M \to \infty$.

While the above result is theoretically interesting, an infinite number of receivers can never be realized in practice. For a finite number of receivers, one can use the efficiency coefficient defined below to evaluate the performance of independent decoding against joint decoding.

Definition 1: The efficiency coefficient of the kth user, e_k, is defined as the ratio between the achievable rates with independent decoding and joint decoding.

Let

$$\boldsymbol{R}_k = \boldsymbol{I} + \sum_{i=1, i\neq k}^{P} \sigma_i^2 \boldsymbol{a}_i \boldsymbol{a}_i^H.$$

From (9), one can express the achievable rate with independent decoding as

$$\begin{aligned}R_k &\leq \frac{1}{2}\log(\det(\sigma_k^2 \boldsymbol{a}_k\boldsymbol{a}_k^H + \boldsymbol{R}_k)/\det(\boldsymbol{R}_k)) \\ &= \frac{1}{2}\log(\det(\boldsymbol{I} + \sigma_k^2\boldsymbol{a}_k\boldsymbol{a}_k^H\boldsymbol{R}_k^{-1})) \\ &= \frac{1}{2}\log(1 + \sigma_k^2 \boldsymbol{a}_k^H \boldsymbol{R}_k^{-1}\boldsymbol{a}_k). \qquad (24)\end{aligned}$$

Therefore, the efficiency coefficient is given by

$$e_k = \frac{\log(1+\sigma_k^2\boldsymbol{a}_k^H\boldsymbol{R}_k^{-1}\boldsymbol{a}_k)}{\log(1+\sigma_k^2 M)}.$$

Unfortunately, the above expression does not seem to be amenable to further mathematical simplification. Numerical examples are thus used to graphically illustrate the changes of $e_k(M)$ as a function of the number of receivers M.

Fig. 6(a) and (c) shows the channel capacity for a two-user MAC with only direct paths at 20° and 50°. The solid and dashed lines depict the capacity regions for joint decoding and independent decoding, respectively. Note that in both the low- and high-SNR scenarios, the capacity regions for joint decoding expand with M and become nearly rectangular at $M = 4$, implying that the performance of independent decoding reaches optimum by exploiting the spatial diversity. This is confirmed by Fig. 6(b) and (d), where the changes of the average efficiency coefficient $(e_1+e_2)/2$ are plotted. These plots support the assertion of Theorem 4.

It is interesting to note that the increase of the efficiency coefficient is not monotonic with respect to the number of antennas. Such behaviors can be explained as follows: while the angle between two vectors tends to increase their sizes increase, such a relation is not monotonic. Therefore, the efficiency coefficient, which is essentially determined by the orthogonality among array response vectors, may actually decrease as the number of antennas increase.

A noticeable difference between the the low-SNR and the high-SNR cases is the improvement of the efficiency coefficients between a one-receiver system and a two-receiver system. From $M = 1$ to $M = 2$, a significantly larger gain is observed when the SNR is higher. This can be explained as follows. When the SNR is high, the dominant interference in independent decoding comes from other users. Unlike spatially white noise, such interference is *directional* in general, which means that it can be effectively suppressed by exploiting the spatial diversity. Therefore, one additional receiver can substantially increase the capacity of independent decoding. On the other hand, the dominant interference in the low-SNR case is white background noise and the gain of SDMA as a result of coherent combining is only a linear function of the number of receivers.

VI. Practical Issues

In this section, we address some practical issues in an SDMA system, using the results obtain from previous sections. Due to the computational complexity, joint decoding is usually prohibited in real applications; independent decoding is more feasible to implement. Our discussion will thus be mainly on the independent decoding scheme.

A. Optimum Projection (Combining)

Since conventional independent decoding is often performed on a one-dimensional sequence, the M outputs from the antenna array are usually combined by a so-called projection

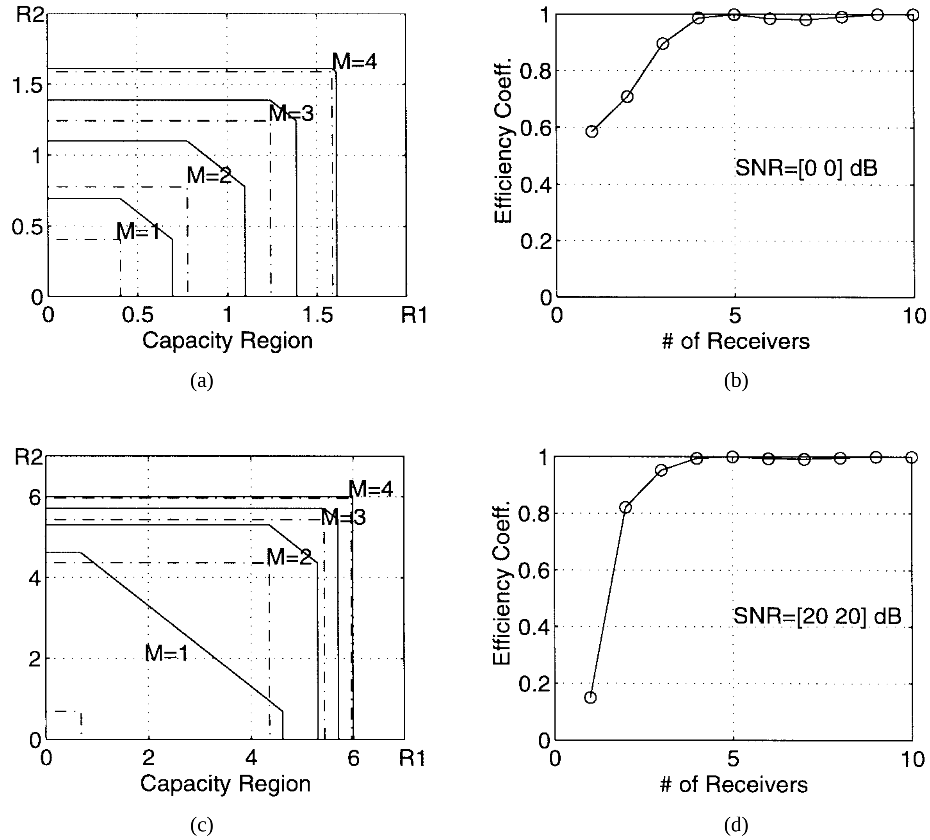

Fig. 6. Capacity region versus number of receivers.

vector $\boldsymbol{w}_k$ before decoding

$$y_k(t) = \boldsymbol{w}_k^H \boldsymbol{x}(t) = \boldsymbol{w}_k^H \boldsymbol{a}_k s_k(t) + \boldsymbol{w}_k^H \left(\boldsymbol{n}(t) + \sum_{i \neq k}^{P} \boldsymbol{a}_i s_i(t) \right), \quad k = 1, \cdots, P. \tag{25}$$

The resulting signal has a covariance of

$$E(y_k(t) y_k^*(t)) = \sigma_k^2 |\boldsymbol{w}_k^H \boldsymbol{a}_k|^2 + \boldsymbol{w}_k^H \boldsymbol{R}_k \boldsymbol{w}_k.$$

Since all formulas were derived based on a multi-input multi-output model, it is thus important to know if the additional combining operation will alter the achievable rate for each user. The answer to this question could be critical to the interface between a conventional single-receiver system and an SDMA system.

We show in the following that, for independent decoding, projection using the Wiener projector will not affect the achievable rate of any user. Although this does not imply that $\boldsymbol{x}(t)$ or $y_k(t)$ are functionally identical, the amount of information of $s_k(t)$ in them is the same.

The Wiener solution that maximizes the signal-to-interference ratio (SIR) of $y_k(t)$ for the kth user is formulated as follows:

$$\arg \max_{\boldsymbol{w}_k} (\text{SIR}_k) = \arg \max_{\boldsymbol{w}_k} \left(\frac{\sigma_k^2 |\boldsymbol{w}_k^H \boldsymbol{a}_k|^2}{\boldsymbol{w}_k^H \boldsymbol{R}_k \boldsymbol{w}} \right) \tag{26}$$

i.e., maximizing SIR_k with respect to $\boldsymbol{w}_k$.

The optimal $\boldsymbol{w}_k$ is given by

$$\begin{aligned} \boldsymbol{w}_k &= \lambda \boldsymbol{R}_k^{-1} \boldsymbol{a}_k \\ \text{SIR}_k &= \sigma_k^2 \boldsymbol{a}_k^H \boldsymbol{R}_k^{-1} \boldsymbol{a}_k \end{aligned} \tag{27}$$

where $\lambda = 1/(\boldsymbol{a}_k^H \boldsymbol{R}_k^{-1} \boldsymbol{a}_k)$.

It can be easily shown from (25) that the achievable rate by performing independent decoding on $y_k(t)$ is given by

$$R_k \leq \frac{1}{2} \log (1 + \text{SIR}_k) = \frac{1}{2} \log (1 + \sigma_k^2 \boldsymbol{a}_k^H \boldsymbol{R}_k^{-1} \boldsymbol{a}_k) \tag{28}$$

which is identical to the achievable rate of independent decoding given in (24). This result leads to the following theorem.

Theorem 5: For independent decoding, the projection using the combining vector does not change the users' achievable rates.

A. Power Control

For independent decoding, the MAC is an interfering channel, which means that each user is an interference source to the others. An important practical issue is power control among the users (i.e., the near–far problem). Clearly, any change of the power level from one user may affect the achievable rates of the others. To maintain certain transmission quality, one wants to balance users' achievable rates, i.e., keep them to be as stable as possible. However, power variations are inevitable in a wireless system. In the following, we study the sensitivity of R_k to power changes of other users in an SDMA system. Since in independent decoding, R_k is proportional to the output

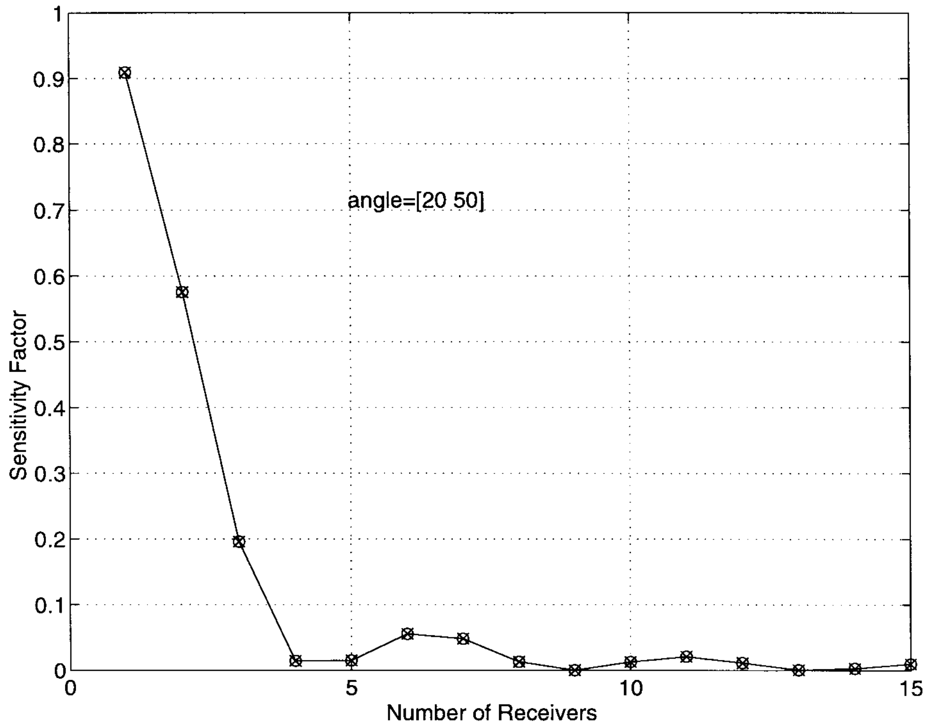

Fig. 7. Power sensitivity versus number of receivers.

$\mathrm{SIR}_k = \sigma_k^2 \boldsymbol{a}_k^H \boldsymbol{R}_k^{-1} \boldsymbol{a}_k$, as shown in (28), it is thus equivalent to examine the sensitivity of SIR_k. In the previous sections, we proved that with sufficient receivers, an SDMA system can provide nearly orthogonal spatial channels for all users, and hence eliminates the co-channel interference. Therefore, one can expect the near–far problem to be mitigated by employing multiple receivers. To quantify the degree of alleviation, we express perturbation of SIR_k in terms of power changes of the other users

$$\Delta \mathrm{SIR}_k = \sum_{j \neq k}^{P} \frac{\partial\, \mathrm{SIR}_k}{\partial \sigma_j^2} \Delta \sigma_j^2. \tag{29}$$

To calculate $\partial\, \mathrm{SIR}_k / \partial \sigma_j^2$, define

$$\boldsymbol{R}_{kj} = \boldsymbol{I} + \sum_{i \neq k,j}^{P} \sigma_i^2 \boldsymbol{a}_i \boldsymbol{a}_i^H$$

and use the matrix inverse lemma

$$\boldsymbol{R}_k^{-1} = \boldsymbol{R}_{kj}^{-1} - \frac{\sigma_j^2 \boldsymbol{R}_{kj}^{-1} \boldsymbol{a}_j \boldsymbol{a}_j^H \boldsymbol{R}_{kj}^{-1}}{1 + \sigma_j^2 \boldsymbol{a}_j^H \boldsymbol{R}_{kj}^{-1} \boldsymbol{a}_j}. \tag{30}$$

Substitute (30) into (27) and note that $\boldsymbol{R}_{kj}$ is not a function of σ_j^2, we obtain

$$\left. \frac{\partial\, \mathrm{SIR}_k}{\partial \sigma_j^2} \right|_{j \neq k} = - \frac{\sigma_k^2 |\boldsymbol{a}_k^H \boldsymbol{R}_{kj}^{-1} \boldsymbol{a}_j|^2}{(1 + \sigma_j^2 \boldsymbol{a}_j^H \boldsymbol{R}_{kj}^{-1} \boldsymbol{a}_j)^2}. \tag{31}$$

Since SIR's are different for systems with different number of receivers, it is only meaningful to express (29) in terms of relative perturbation

$$\frac{\Delta\, \mathrm{SIR}_k}{\mathrm{SIR}_k} = \sum_{j \neq k}^{P} \underbrace{\left(\frac{\sigma_j^2}{\mathrm{SIR}_k} \frac{\partial\, \mathrm{SIR}_k}{\partial \sigma_j^2} \right)}_{d_{k,j}} \frac{\Delta \sigma_j^2}{\sigma_j^2}. \tag{32}$$

We use $(\Sigma_{j \neq k}^{P} d_{k,j}^2)^{1/2}$ as an overall power sensitivity measure of the kth user. Clearly, low sensitivity which means high resistance to power variations is desirable.

An example is given in Fig. 7 which plots the power sensitivity versus the number of receivers using the same setup as in Fig. 6(a). Similar to the situations in Fig. 6, the sensitivity reduces dramatically by adding a few receivers to a single-receiver system. After the $M \geq 4$, the array response vectors become nearly orthogonal and cochannel interference disappears. In other words, with sufficient receivers, each user in the SDMA system becomes immune to other users' power variations.

VII. Conclusion

In this paper, we have investigated the uplink channel capacity of an SDMA system by modeling it as a set of parallel multiple-access Gaussian channels. The capacity regions for joint decoding and independent decoding have been derived, based on which we identified their inner and outer boundaries. We proved that exploitation of the spatial diversity among the users allows nearly optimum performance to be achieved by independent decoding, and that the achievable rate for independent decoding would not be affected by optimum combining. We have also demonstrated that the use of multiple receivers in SDMA can dramatically increase the system's immunity to power variations.

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