An Analysis on Uplink OFDMA Optimality

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Abstract

Motivated by the increasing popularity of OFDMA, this paper studies the sum-rate optimality of OFDMA as an uplink multicarrier multiple-access scheme. We address the relationship between OFDMA and the optimal multicarrier multiple-access schemes. More specifically, we are interested in answering the following three questions: (i) what are the conditions under which OFDMA is sum-rate optimal? (ii) what is the probability of OFDMA being sum-rate optimal? and (iii) in the case OFDMA is suboptimal, what is the performance gap between OFDMA and the optimal multicarrier multiple access solution? Within a generic discrete multicarrier uplink framework, we present the necessary and sufficient conditions for OFDMA optimality and derive the probabilities of these conditions for both low and high SNR regions. Our results show that the number of shared subchannels under the optimal solution should be less than the number of total users, thus the performance gap between the OFDMA scheme and the optimal solution is negligible when the number of subchannels is large.

Index Terms–OFDMA, uplink, optimality, multicarrier

I. Introduction

Orthogonal frequency division multiple access (OFDMA) has emerged as one of the prime multiple-access schemes for broadband wireless networks (e.g., IEEE 802.16 Mobile WiMAX, IEEE 802.20 and 3G LTE, etc.). As a special case of multicarrier multiple-access schemes, OFDMA exclusively assigns each subchannel to no more than one user, eliminating intra-cell interference (ICI). For fixed or portable applications where the frequency selective channels are slowly varying, an intrinsic advantage of OFDMA is its capability to exploit the so-called multiuser diversity through subchannel allocation [1] [2]. Furthermore, OFDMA has the merit of easy decoding at the receiver side due to the absence of ICI. Other advantages of OFDMA include finer granularity and better link budget in uplink communications.

An important performance measure in broadband multiple-access is the sum-rate capacity. Despite the successful adoption of OFDMA in practical wireless systems, a fundamental question, namely, whether or not OFDMA is the sum-rate optimal multiple-access scheme, is yet to be thoroughly investigated.

The input optimization (power allocation) problem for the general multicarrier multiple-access channel has been studied in the literature for several cases. The capacity region of a Gaussian multiple-access channel with ISI was characterized by Cheng and Verdu [3]. The optimal power allocation over time for i.i.d. fading uplink channels has been investigated by Tse and Hanly [4]. In [5], the author proposed an efficient iterative water filling method to numerically compute the optimal power allocation for uplink MIMO channels. In all cases, the channel information is assumed to be known. For OFDMA, the prior work has been focused on the subchannel allocation/power loading problem which can be cast into a classic multiuser waterfilling framework. The topic has been the subject of many studies and a number of implementation schemes can be found in the literature, see [6]-[13] and references therein. Our interest here is not on the algorithmic aspect of OFDMA subchannel loading. Rather, we attempt to address the problem at a more fundamental level: what is the relationship between OFDMA and the optimal multiple-access scheme? We concentrate on the whether OFDMA, as a constrained multicarrier multiple-access scheme, is sum-rate optimal from an information theoretic prospective. And if not, the objective is to characterize the nature of the sum-rate loss due to limiting at most one user on each subchannel.

Given the importance of OFDMA, the amount of academic research on OFDMA optimality is still quite limited (relative to 3G/CDMA) to date. In [14], Jang and Lee proved the sum-rate optimality of OFDMA in downlink with adaptive QAM modulation and independent decoding. Li and Liu generalized the results to downlink OFDMA with MIMO [15], again assuming only independent decoding. Note that the sum-rate achieved by independent decoding is always inferior to the optimal sum-capacity, which can be achieved by superposition coding and successive interference cancellation [16]. More recently, Michel and Wunder presented results on OFDMA optimality for downlink MIMO channels from an information-theoretic viewpoint [17]. In [3] and [18], it is stated that in order to maximize the uplink sum-capacity, only one user should transmit at any given frequency over the entire spectrum. However this result is only valid for continuous frequency selective channels and can not be applied to discrete multicarrier multiple-access systems with a finite number of subchannels (as in OFDMA). As a matter of fact, as the main result of this paper shows, OFDMA is suboptimal with non-trivial probabilities in many scenarios. Other uplink OFDMA optimality studies can also be found in [19]-[21]. None of these results however, provide a quantitative analysis with regard to OFDMA optimality with a finite number of subchannels.

In this paper, we consider a generic discrete multicarrier multiuser uplink channel and derive the necessary and sufficient conditions under which OFDMA is sum-rate optimal. We further derive the probabilities of these conditions for both low and high SNR regions. Another contribution of this paper is that we show the number of shared subchannels under the optimal solution should be less than the number of total users. As a result, the performance gap between the OFDMA scheme and the optimal solution becomes negligible when the number of subchannels is large. On the other hand, it is difficult to completely quantify the performance gap between OFDMA and the optimal solution under arbitrary configurations (e.g., the number of subchannels, the number of users, SINR, etc.) and the topic remains an area of future research.

The rest of the paper is organized as follows. Section II describes a generic uplink multicarrier multiple-access model. The sum-rate optimization problem is formulated accordingly to determine the optimal power allocation (subject to individual power constraints) for such systems. In section III, we first present a theorem that provides important insight on the optimal multicarrier solution. The results allow us to derive the necessary and sufficient conditions under which OFDMA is indeed optimal. We then analyze the optimality of OFDMA for both low and high SNR regions and calculate the probabilities. In the case OFDMA is suboptimal, a theorem is presented to address the performance gap between the OFDMA scheme and the optimal solution. Section IV provides numerical results to validate the analyses in Section III for subchannels with both strong and weak correlations. The paper is then concluded in Section V.

II. System Model

We invoke the following assumptions for the rest of the paper: (i) perfect knowledge of the channel state information (CSI) for both the transmitter and receiver; (ii) either static or fading channels, in both cases, the channel coefficients are random; (iii) independent channels among users; (iv) for each individual user, the channel gains over frequency subchannels are correlated.

Since the transmitter has the perfect CSI, it shall dynamically assign subchannels and allocate power to optimize the performance, i.e., we are considering channel-state-dependent multicarrier systems.

We use the following notation conventions throughout the paper:

K	the total number of users
N	the total number of subchannels
h_{kn}	the channel gain associated with subchannel n and user k
p_{kn}	the transmission power of user k on subchannel n
p_k	user k's total transmitting power in uplink channel
B	the total system bandwidth
B_n	subchannel bandwidth
B_c	coherent bandwidth
P_o	the probability of OFDMA being sum-rate optimal
$\max\limits_x \{f(x)\}$	maximum value of $f(x)$ maximized over all x
$\arg\max\limits_x \{f(x)\}$	value of x that maximizes the function $f(x)$

For the generic multicarrier multiple-access, all the users can transmit on all the subchannels subject to individual peak power constraints $p_k : \sum\limits_{n=1}^{N} p_{kn} \leq p_k$, where $1 \leq k \leq K$. We assume the total bandwidth B is fixed and the subchannel bandwidth B_N decreases linearly with N, i.e. $B_N \sim B/N$ [16].

The non-interfering parallel subchannels [22, pp. 182] can be created using FFT and IFFT operations [23]. Let x_{kn} be the information-bearing signal from user k on subchannel n. The received signal on subchannel n can be expressed as:

$$y_n = \left(\sum_{k=1}^{K} x_{kn} \sqrt{p_{kn}} h_{kn} \right) + \upsilon_n \qquad (1)$$

where υ_n is the AWGN noise with variance N_0. For each fading state (channel realization), the capacity region on subchannel n is given in [24, Chap. 14.1] (assuming $E\left(\|x_{kn}\|^2\right) = 1$; and

x_{jn}, x_{kn} are independent zero-mean random variables):

$$c_n = (R_{1n}, ..., R_{Kn}) : \sum_{k \in S} R_{kn} \le B_N \log_2 \left(1 + \frac{\sum\limits_{k \in S} p_{kn} \left\| h_{kn} \right\|^2}{N_0 B_N} \right)$$
$$\forall\ S \subset \{1, 2, ...K\}\,. \tag{2}$$

Equation (2) indicates that the sum of rates on subchannel n for any subset of the K users is upper limited by the capacity of a "super user" with received power equals to the sum of received powers associated with the particular user subset on this subchannel. The capacity region in (2) can be achieved by superposition coding and successive interference cancellation [16, pp.472-3]. Since information is delivered through parallel subchannels in multicarrier, the total capacity region over all subchannels and all users is then [22] [24, Chap. 10.4] [25]:

$$C\left([p_{kn}]_{K \times N}\right) = (R_1, ..., R_K) : \sum_{k \in S} R_k \le B_N \sum_{n=1}^{N} \log_2 \left(1 + \frac{\sum\limits_{k \in S} p_{kn} \left\| h_{kn} \right\|^2}{N_0 B_N} \right)$$
$$\forall\ S \subset \{1, 2, ...K\}. \tag{3}$$

We choose the *sum-of-rate capacity*, C_{sum}, as the figure of merit. To maximize the sum-capacity, the transmission power needs to be distributed optimally under the individual peak power constraints. The following optimization problem is formulated accordingly:

$$\arg\max_{p_{kn}} C_{sum} = B_N \sum_{n=1}^{N} \log_2 \left(1 + \frac{\sum\limits_{k=1}^{K} p_{kn} \left\| h_{kn} \right\|^2}{N_0 B_N} \right) \tag{4}$$
$$s.t.\ \sum_{n=1}^{N} p_{kn} \le p_k,\ p_{kn} \ge 0$$
$$\forall\ k,\ n, \text{ where } 1 \le k \le K,\ 1 \le n \le N.$$

The optimization problem (*4*) is also widely used in frequency-flat fading channels [1] [4] and multiuser waterfilling with informed static MIMO channels [5] [26] [27].

Remark 1: With perfect CSI at both the transmitter and receiver, the ergodic sum-capacity for fading channels is defined under individual user's average power constraints (energy constraints) *as [16] [22]:*

$$\arg\max_{p_{kn}} C_{sum} = \mathbf{E_H}\left[B_N \sum_{n=1}^{N} \log_2\left(1 + \frac{\sum_{k=1}^{K} p_{kn}\left\|h_{kn}\right\|^2}{N_0 B_N}\right)\right] \tag{5}$$

$$s.t.\ \mathbf{E_H}\left[\sum_{n=1}^{N} p_{kn}(\mathbf{H})\right] \le \bar{p}_k,\ \ p_{kn}(\mathbf{H}) \ge 0$$

$$where\ \mathbf{H} = (\overrightarrow{\boldsymbol{h}}_1, \overrightarrow{\boldsymbol{h}}_2, ...\overrightarrow{\boldsymbol{h}}_K)^T \text{ is the channel realization}$$

However, in practical uplink scenarios, users (terminal stations) are subject to 'instantaneous' peak-power constraints [22], i.e., the instantaneous maximum transmitting power is dictated by the power amplifier of the terminals. Under these peak-power constraints, the maximum power for each channel realization (fading state) is fixed, thus the optimization problem (5) reduces to (4).

Remark 2: Joint decoding is needed to achieve the sum-capacity in (4). A suboptimal but simpler scheme is independent decoding. With independent decoding, the objective function in (4) becomes

$$C_{sum} = B_N \sum_{n=1}^{N}\sum_{k=1}^{K} \log_2\left(1 + \frac{p_{kn}\left\|h_{kn}\right\|^2}{\sum_{\substack{i=1\\ i\neq k}}^{K} p_{in}\left\|h_{in}\right\|^2 + N_0 B_N}\right) \tag{6}$$

The optimality analyses in the following sections can be applied to independent decoding as well.

In the remainder of this paper, we shall refer to the solution to (4) as the sum-rate optimal multicarrier multiple-access scheme, or simply, *the optimal solution*.

Define S_n as the active user set on subchannel n: $\{i \in S_n \text{ iff } p_{in} > 0\}$. Clearly, S_n ($S_n \subset \{1, 2, ...K\}$) is a subset of the K users. As a special multiple-access scheme, OFDMA can be defined mathematically as

$$|S_n| \le 1, \text{ for } n = 1, ..., N$$

That is, each subchannel is exclusively assigned to no more than one user. Since subchannel allocations vary with channel state, we are considering dynamic OFDMA instead of static OFDMA. For discrete multicarrier systems, the optimal solution scheme may or may not be

OFDMA. The rest of the paper focuses on the relationship between OFDMA and the optimal solution. Our primary interest is to determine (i) under what conditions is OFDMA the optimal multiple-access scheme; (ii) what is the probability for these conditions, and (iii) in the case OFDMA is not optimal, what is the performance gap between OFDMA and the optimal solution.

III. Analysis of OFDMA Optimality

In this section, we first provide a general theorem which asserts the optimal solution of problem in (4) without completely solving it. The results nevertheless allow us to derive the necessary and sufficient conditions under which OFDMA is optimal. Based on these conditions, we calculate P_o (the probability of OFDMA being optimal) for the small and the large SNR regions respectively in Subsections III.B and III.C. In the case OFDMA is not optimal, we further show in subsection III.D that the number of shared subchannels should be less than K in the optimal solution, thus the performance gap between OFDMA scheme and the optimal solution is negligible when $N >> K$.

Theorem 1: Referring to (4), the sum-rate capacity C_{sum} is maximized only if the following conditions are satisfied:

(i) *Subchannel n is utilized by at least one user if and only if $\max_k \left\{ \frac{\|h_{kn}\|^2}{\lambda_k} \right\} > N_0$; and the transmitting user(s) must have the highest weighted power gain $\max_k \left\{ \frac{\|h_{kn}\|^2}{\lambda_k} \right\}$, where λ_k is the Lagrange multiplier that enforces the power constraint on the k^{th} user .*

(ii) *If the optimal solution suggests OFDMA, each user allocates its power within its allocated subchannels by water filling with water level at $\frac{1}{\lambda_k}$ [24, Chap. 10.4].*

The proof uses the following Lemma:

Lemma 1: Karush-Kuhn-Tucker conditions (First-Order Necessary Conditions) [28, Theorem 12.1]: For a general optimization problem

$$\begin{aligned} &\min_{\mathbf{x} \in R^n} f(x) && (7) \\ s.t. \quad &h_i(\mathbf{x}) = 0, \ i \in \mathcal{E} \\ &h_i(\mathbf{x}) \geq 0, \ i \in \mathcal{I} \end{aligned}$$

where $f(\mathbf{x})$ and $h_i(\mathbf{x})$ are all smooth, real-valued functions on a subset of R^n, I and E are two *finite sets of indices. The Lagrange of (7) is defined as*

$$L(\mathbf{x}, \boldsymbol{\mu}) = f(\mathbf{x}) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \mu_i h_i(\mathbf{x}).$$

If $\mathbf{x}^*$ *is a local solution of (7) and the linear independence constraint qualification (LICQ) holds at* $\mathbf{x}^*$*, then there is a Lagrange multiplier vector* $\boldsymbol{\mu}^*$ *with components* $\mu_i^*, i \in E \cup I$*, such that the following conditions are satisfied at (*$\mathbf{x}^*, \boldsymbol{\mu}^*$*)*

$$\nabla_{\mathbf{x}} L(\mathbf{x}^*, \boldsymbol{\mu}^*) = 0 \tag{8}$$

$$h_i(\mathbf{x}^*) = 0, \ i \in \mathcal{E} \tag{9}$$

$$h_i(\mathbf{x}^*) \geq 0, \ i \in \mathcal{I} \tag{10}$$

$$\boldsymbol{\mu}^* \geq 0, \ i \in \mathcal{I} \tag{11}$$

$$\mu_i^* h_i(\mathbf{x}^*) = 0, \ i \in \mathcal{E} \cup \mathcal{I} \tag{12}$$

Furthermore, if (7) is an convex optimization problem (the objective function $f(\mathbf{x})$ *is convex and the feasible region is also convex), then conditions from (8) to (12) are also sufficient conditions to determine the global optimal solution* $\mathbf{x}^*$ *[29].*

We can prove that $-C_{sum}$ is a convex function with respect to p_{kn} (See appendix A). Since both the equality and inequality constraints are linear and the LICQ condition holds as well, the optimization problem (4) can be cast into a classic convex programming problem [29]. The Karush-Kuhn-Tucker (KKT) conditions can be used to determine the optimal solution. In order to do so, we first construct the Lagrangian function of (4) as

$$\begin{aligned} L(p, \ \lambda) = & -B_N \sum_{n=1}^{N} \log_2 \left(1 + \frac{\sum_{k=1}^{K} p_{kn} \left\| h_{kn} \right\|^2}{N_0 B_N} \right) \\ & + \sum_{k=1}^{K} \lambda_k (\sum_{n=1}^{N} p_{kn} - p_k) - \sum_{k=1}^{K} \sum_{n=1}^{N} \lambda_{kn} p_{kn} \end{aligned} \tag{13}$$

The KKT conditions state that (p^*, λ^*) is the optimal solution if and only if the pair satisfies the following conditions:

$$\frac{B_N \left\|h_{kn}\right\|^2}{\sum_{k=1}^{K} \left\|h_{kn}\right\|^2 p_{kn}^* + N_0 B_N} + \lambda_{kn}^* = \lambda_k^* \tag{14}$$

$$\sum_{n=1}^{N} p_{kn}^* - p_k = 0, \tag{15}$$

$$p_{kn}^* \geqslant 0 \tag{16}$$

$$\lambda_{kn}^* p_{kn}^* = 0 \tag{17}$$

$$\lambda_{kn}^* \geq 0 \tag{18}$$

where $1 \leq k \leq K$ and $1 \leq n \leq N$.

Remark 3: For optimal power allocation with independent decoding, we can arrive at the same optimal conditions as in (14) to (18). However, these conditions are not sufficient conditions because the objective function (6) is neither convex nor concave.

Using the above results, the proof of the first claim of Theorem 1 can be found in appendix B and the second claim is quite straightforward.

From Theorem 1, we conclude that, under the optimal solution, (i) no user should transmit on subchannel n iff $\max_k \left\{\frac{\|h_{kn}\|^2}{\lambda_k}\right\} \leq N_0$; (ii) multiple users may transmit on the same subchannel n iff they have the same weighted power gain $\max_k \left\{\frac{\|h_{kn}\|^2}{\lambda_k}\right\} > N_0$. In this case OFDMA is no longer the optimal solution. One may argue that the probability of multiple users having identical weighted gains is zero due to random channels, thus OFDMA is indeed the optimal solution. While intuitively true, such an argument is invalid for finite discrete multiple-access channels. A counter example is provided for illustration.

Example 1: Define $\alpha_1 = \frac{\|h_{11}\|^2}{\|h_{21}\|^2}$, $\alpha_2 = \frac{\|h_{12}\|^2}{\|h_{22}\|^2}$, $n_{kn} = \frac{B_N N_0}{\|h_{kn}\|^2}$ *in a two-user, two-subchannel scenario. Part of the closed-form analytical solutions of (4) is summarized in Table 1 for different channel conditions.*

From Table 1, we clearly observe that OFDMA may or may not be optimal, depending on channel gains $\{h_{kn}\}$ and user's individual transmitting power $\{p_k\}$. Under the scenario where $\alpha_1 \leq \alpha_2$ and $|\alpha_2 p_1 + n_{22} - n_{21}| < p_2$, the optimal solution yields $\frac{h_{12}}{\lambda_1} = \frac{1}{2}(h_{12}p_1 + h_{22}p_2 + h_{22}n_{21} - N_0) = \frac{h_{22}}{\lambda_2}$. That is, with probability 1 the two users will have the same weighted power gain on subchannel 2. On the other hand, when $\alpha_2 p_1 + n_{22} - n_{21} \geq p_2$ and $\frac{p_2}{\alpha_1} + n_{11} - n_{12} \geq p_1$, we have $\frac{h_{11}}{\lambda_1} \leq \frac{h_{21}}{\lambda_2}$ on subchannel 1 and $\frac{h_{12}}{\lambda_1} \geq \frac{h_{22}}{\lambda_2}$ on subchannel 2. The probability that two users have the same weighted power gain is zero. Given random channel realizations, clearly there is a non-trivial probability that OFDMA is not the optimal solution.

A. Necessary and sufficient conditions

Next, we derive the necessary and sufficient conditions under which OFDMA is optimal.

Let us define a specific OFDMA subchannel allocation scheme as $\Phi = \{\phi_1, \phi_2, ...\phi_K\}$, where ϕ_k is the set of subchannels assigned to user k [6]-[13]. Denote $I_k = |\phi_k|$ be the number of subchannels assigned to user k. Since $I_k \geq 1$, OFDMA requires $N \geq K$. Under the OFDMA subchannel allocation Φ, we can calculate its power loading $\Psi = (p^*, \lambda^*)$ using single user water filling [24]. Here, we are interested to know whether this specific OFDMA power allocation scheme is optimal to (4).

Using Theorem 1, we arrive at the following results:

Proposition 1: The necessary and sufficient conditions for OFDMA subchannel allocation Φ (power allocation Ψ) to be the optimal solution to (4) are:

$$\begin{aligned} \frac{\|h_{kn}\|^2}{\lambda_k} = \max_i \left\{\frac{\|h_{in}\|^2}{\lambda_i}\right\}, \quad &\text{for } S_n = \{k\} \\ \max_i \left\{\frac{\|h_{in}\|^2}{\lambda_i}\right\} \leq N_0, \quad &\text{for } S_n = \emptyset \\ \text{where } n = 1, 2...N& \end{aligned} \tag{19}$$

and the weight $\frac{1}{\lambda_k}$ is user k's single user water filling level.

Proof: Necessary conditions: Assume the OFDMA power allocation scheme Ψ is the optimal solution to (4). In the case $|S_n| = 1$, part (i) of Theorem 1 asserts that the active user k must have the highest weighted power gain on subchannel n for any $n \in \phi_k$; in the case $|S_n| = 0$, part (i) of Theorem 1 requires $\max_i \left\{\frac{\|h_{in}\|^2}{\lambda_i}\right\} \leq N_0$. Thus, (19) is a necessary condition for Ψ to be optimal.

Sufficient condition: With OFDMA power allocation Ψ, we can verify that (p^*, λ^*) satisfies the KKT conditions (14) to (18) if (19) is true. By lemma 1, $\Psi = (p^*, \lambda^*)$ is the optimal solution to (4) and Φ is the optimal subchannel allocation. Therefore, (19) is also a sufficient condition for Ψ to be the optimal allocation. ■

If a specific OFDMA subchannel allocation is optimal, for any active subchannel, whoever experiences the largest weighted power gain should be allocated that subchannel. Now that we know OFDMA is not always the optimal solution, our next interest is in P_O, the probability that OFDMA is indeed the sum-rate optimal. The probability of conditions (19) defines P_Ψ, i.e., the probability that the OFDMA allocation Ψ is the optimal allocation. Note that P_O is the sum of P_Ψ for all possible $\{\Psi\}$. We can numerically determine P_O as long as the probability density function (PDF) of the channel gain h_{kn} is given. However, the computation can be prohibitively expensive when K and N are large. For tractability, we provide analyses on two extreme scenarios: (i) the low signal to noise ratio (SNR) case and (ii) the high SNR case. We will show in Section IV that our analyses for low and high SNR regions provide the upper and lower bounds for P_o in moderate SNR region respectively.

B. Low SNR Case

In low SNR region, we assume

$$\sum_{k=1}^{K} \|h_{kn}\|^2 p_{kn} \ll B_N N_0, \quad for \quad \forall\, n \tag{20}$$

Then (14) becomes

$$\frac{\|h_{kn}\|^2}{N_0} + \lambda_{kn} = \lambda_k,\ 1 \le k \le K,\ 1 \le n \le N \tag{21}$$

The following theorem provides the solution to (4) in low SNR scenario.

Theorem 2: (i) C_{sum} is maximized if and only if each user allocates all its power to only one subchannel and the assigned subchannel to user k has the highest channel gain $\|h_{kn}\|$ over all n; (ii) $P_O = \frac{N!}{(N-K)!N^K}$.

Proof: We use reduction to absurdity to prove the first claim: assume in the optimal solution user k transmits on both subchannel n_1 and n_2 i.e. $p_{kn_1} \ne 0, p_{kn_2} \ne 0$. From Equation (17), we have

$$\lambda_{kn_1} = \lambda_{kn_2} = 0$$

Plugging above result into (21) yields

$$\frac{\|h_{kn_1}\|^2}{N_0} = \lambda_k = \frac{\left\|h_{kn_2}\right\|^2}{N_0}$$

That is, $\|h_{kn_1}\| = \|h_{kn_2}\|$. However, the probability of $\{\|h_{kn_1}\| = \|h_{kn_2}\|\}$ is zero because of random channel gains. Thus in the optimal solution each user should allocate all its power to only one subchannel.

Next, let us assume that user k allocates its power p_k to subchannel n_1 ($p_{kn_1} = p_k$). If $n_1 \neq \arg\max_n(\|h_{kn}\|)$, then there exists a subchannel n_2 such that $p_{kn_2} = 0$ and $\|h_{kn_1}\| < \|h_{kn_2}\|$. From (17) and (18), we have $\lambda_{kn_1} = 0$ and $\lambda_{kn_2} \geq 0$. Plugging $\lambda_{kn_1} = 0$ into (21) yields

$$\frac{\|h_{kn_1}\|^2}{N_0} = \lambda_k \tag{22}$$

Plugging $\lambda_{kn_2} \geq 0$ into (21) yields

$$\frac{\|h_{kn_2}\|^2}{N_0} + \lambda_{kn_2} = \lambda_k \tag{23}$$

From (22) and (23), we have $\|h_{kn_1}\| \geqslant \|h_{kn_2}\|$, which contradicts with the fact that $\|h_{kn_1}\| < \|h_{kn_2}\|$. As a result, we have proved that

$$p^*_{kn} = \begin{cases} p_k, & n = \arg\max_n(\|h_{kn}\|) \\ 0, & otherwise \end{cases} \tag{24}$$

i.e., each user must allocate all its power to the subchannel with the maximum gain.

For part (ii), notice that P_O simply is the probability that the subchannels of highest gains are different for different users, therefore

$$P_O = \begin{pmatrix} N \\ 1 \end{pmatrix} \dots \begin{pmatrix} N-K+1 \\ 1 \end{pmatrix} / \begin{pmatrix} N \\ 1 \end{pmatrix}^K = \frac{N!}{(N-K)!N^K} \tag{25}$$

■

Fig. 1 plots the P_O versus N with different K. From this figure, we can see that there is a significant chance that OFDMA is optimal under low SNR. Further, the P_O increases with N and decreases with K.

C. High SNR Case

For the high SNR case, we assume

$$\textstyle\sum_{i=1}^K \|h_{in}\|^2 p_{in} \gg N_0 B_N, \quad \forall \text{ utilized subchannel } n \tag{26}$$

Then Condition (14) for utilized subchannel reduces to

$$\frac{B_N \|h_{kn}\|^2}{\sum_{i=1}^K \|h_{in}\|^2 p_{in}} + \lambda_{kn} = \lambda_k, \tag{27}$$

The following theorem states the solution to (4) in high SNR case.

Theorem 3: (i) C_{sum} is maximized only if every subchannel is utilized, i.e. there is no empty subchannel; (ii) Assuming $\|h_{kn}\|$s are i.i.d. R.V. with Rayleigh distribution, P_O is given by

$$P_o = \sum_{I_1=1}^{N-(K-1)} \cdots \sum_{I_{K-1}=1}^{N-(I_1+I_2+\ldots+I_{K-2})} \left\{ \begin{pmatrix} N \\ I_1 \end{pmatrix} \cdots \begin{pmatrix} N - \sum_{j=1}^{K-2} I_j \\ I_{K-1} \end{pmatrix} P_\Psi \right\} \tag{28}$$

$$\text{where } P_\Psi = \prod_{k=1}^{K} \left[\int_0^1 \prod_{j \neq k} (1 - x^{\frac{I_j}{I_k} \frac{p_k}{p_j}}) dx \right]^{I_k} \tag{29}$$

Proof: We prove part (i) using reduction to absurdity. Assuming subchannel n_1 is empty, then Condition (14) becomes

$$\frac{B_N \|h_{kn_1}\|^2}{N_0} + \lambda_{kn_1} = \lambda_k, \quad \forall\ k \tag{30}$$

We know that there exists at least one subchannel n_2 for which $p_{kn_2} \neq 0$. Equation (27) for subchannel n_2 becomes

$$\frac{B_N \|h_{kn_2}\|^2}{\sum_{i=1}^{K} \|h_{in_2}\|^2 p_{in_2}} = \lambda_k \tag{31}$$

From (26), (30) and (31), we obtain $\lambda_{kn_1} < 0$, which contradicts with (18). Thus, we have argued that every subchannel should be used by at least one user. Since there is no empty subchannel, each OFDMA power loading Ψ corresponds to only one OFDMA subcarrier allocation Φ.

To calculate P_O, assume that the OFDMA scheme Ψ (Φ) is the optimal solution. From (31), user k's water filling level over ϕ_k is

$$\frac{1}{\lambda_k} = \frac{p_k}{B_N I_k} \tag{32}$$

Condition (19) yields

$$\begin{aligned} \frac{\|h_{1n}\|}{\|h_{in}\|} &\geq \sqrt{\frac{I_1}{I_i}\frac{p_i}{p_1}}, \forall\ i \neq 1,\, n \in \phi_1 \\ &\vdots \\ \frac{\|h_{Kn}\|}{\|h_{in}\|} &\geq \sqrt{\frac{I_K}{I_i}\frac{p_i}{p_K}}, \forall\ i \neq K,\, n \in \phi_K \end{aligned} \tag{33}$$

Appendix C derives the probability of condition (33) for Rayleigh channels

$$P_\Psi = \left[\int_0^1 \prod_{j \neq 1} (1 - x^{\frac{I_j p_1}{I_1 p_j}}) dx \right]^{I_1} \cdots \left[\int_0^1 \prod_{j \neq K} (1 - x^{\frac{I_j p_K}{I_K p_j}}) dx \right]^{I_K}$$

For each fixed set of $\{I_1, I_2, ..., I_K\}$, the number of possible Ψs (Φs) is $\begin{pmatrix} N \\ I_1 \end{pmatrix} ... \begin{pmatrix} N - \sum_{j=1}^{K-2} I_j \\ I_{K-1} \end{pmatrix}$. Including all possible sets of $\{I_1, I_2, ..., I_K\}$, P_o is given by

$$P_O = \sum_{I_1=1}^{N-(K-1)} ... \sum_{I_{K-1}=1}^{N-(I_1+I_2+...+I_{K-2})} \left\{ \begin{pmatrix} N \\ I_1 \end{pmatrix} ... \begin{pmatrix} N - \sum_{j=1}^{K-2} I_j \\ I_{K-1} \end{pmatrix} P_\Psi \right\}$$

■

Remark 4: When $K = N$, both Equation (25) and (28) yield the same $P_o = K!/K^K$.

Note that we assume i.i.d channels to simplify the derivation of (29). In reality, the adjacent subcarriers have similar channel gains when subcarriers have strong correlation. One solution is to group these adjacent subcarriers as a single subchannel, as in WiMax IEEE 802.16e. The grouped subchannels are much less correlated and can be approximated by i.i.d analysis. Our numerical results in Section IV validate this approximation.

D. Suboptimal Case

In the case when the optimal solution is not OFDMA, we would like to know what is the performance gap between OFDMA and the optimal solution. The following Theorem provides the answers.

Theorem 4: (i) The total capacity C_{sum} is maximized only if the number of shared subchannels is less than K. (ii) The capacity gap between OFDMA and the optimal solution diminishes when $N >> K$. (iii) OFDMA is optimal when $N \rightarrow \infty$.

The proof of Theorem 4 uses the following definitions and corollaries in [30]:

Definition 1: (i) A network is called a tree if it is connected and contains no elementary circuits but has at least one arc. (ii) An arc set $F \subset A$ is said to form a forest in the network G if and only if no elementary circuits are included in F. (iii) A maximal forest in G is defined to be a forest that is not strictly contained in any other forest of G. (iv) A spanning tree for G is a tree meeting every node of G.

Corollary 1: (i) In a connected network (with at least one arc) a maximal forest is the same thing as a spanning tree. (ii) A connected network is a tree if and only if [the number of arcs] = [the number of nodes] − 1 > 0.

If the optimal solution requires m ($m > 1$) users to transmit on subchannel n, we define a network g_n with node set $u_n \subset \{1, ..., K\}$, where $|u_n| = m$ and each node in u_n represents one of the m users. For any two different nodes (users) $i,\ j \in u_n$, we obtain from Theorem 1 the following constraint

$$\frac{\|h_{in}\|^2}{\lambda_i^*} = \frac{\|h_{jn}\|^2}{\lambda_j^*} \tag{34}$$

Introducing an arc a_{ij} between node i and j to denote the above constraint, then g_n is a connected network with $\begin{pmatrix} m \\ 2 \end{pmatrix}$ different arcs.

By eliminating unnecessary arcs without changing the constraints among all nodes, we can transform g_n into a tree t_n with only $m - 1$ arcs. The idea is illustrated with the following example:

Example 2: For a subchannel n used by three users, Fig. (2) shows the network g_n. We can remove the arc j_{13} without losing the constraint between node 1 and 3. To see this, we have from (34) the constraints $\frac{\|h_{1n}\|^2}{\|h_{2n}\|^2} = \frac{\lambda_1^*}{\lambda_2^*}$ and $\frac{\|h_{2n}\|^2}{\|h_{3n}\|^2} = \frac{\lambda_2^*}{\lambda_3^*}$ for arc j_{12} and j_{23} respectively. Multiplying these two equations yields $\frac{\|h_{1n}\|^2}{\|h_{3n}\|^2} = \frac{\lambda_1^*}{\lambda_3^*}$, which is the same constraint indicated by arc j_{13}. Thus, t_n is equivalent to g_n in Fig. (2).

If OFDMA is suboptimal, there exists r ($r \geq 1$) trees corresponding to r different subchannels used by multiple users. These r trees together form a new network G. Before the proof of Theorem 4, we first prove the following lemma:

Lemma 2: With probability 1, the network G comprising r trees forms a forest.

Proof: G forms a forest if and only if no elementary circuits are included in G [30]. Without loss of generosity, we assume there exists a circuit $c : 1 \longleftrightarrow 2 \longleftrightarrow 3 ... n \longleftrightarrow 1$. Since each tree contains no elementary circuits, the circuit c contains arcs from at least two different trees. Say arc $j_{12} \in t_1$, $j_{23} \in t_2$, and $j_{1n} \in t_x$, where $t_1 \neq t_2$. From (34), we have

$$\frac{\|h_{11}\|^2}{\|h_{21}\|^2} = \frac{\lambda_1^*}{\lambda_2^*} = \frac{\|h_{1x}\|^2}{\|h_{nx}\|^2} ... \frac{\|h_{32}\|^2}{\|h_{22}\|^2} \tag{35}$$

Since $\|h_{kn}\|$s are random variables with continuous PDF, Equation (35) holds with zero probability. Thus, with probability 1, the network G comprising all possible trees contains no elementary circuits, i.e., G forms a forest. ■

Based on the above definitions, corollaries and Lemma 2, we can easily prove Theorem 4:

Proof: [Proof of Theorem 4] From Lemma 2, we know G is a forest. Furthermore, when the number of trees reaches maximum, by Definition 1.iii, G becomes a maximal forest. This is true because we can always add an arc (tree) to G if G is not a maximal forest. By Corollary 1 and Definition 1.iv, the maximal forest G is a spanning tree and $[\text{the number of arcs}] = K - 1$. Note that a tree has at least one arc. Therefore, we have proved that the maximum number of trees $r_{\max} \leq K - 1$. That is, the number of subchannels that can be assigned to multiple users is less than K.

For the r subchannels used by multiple users, from the second claim of Theorem 1, we have

$$\sum_{k=1}^{K} \|h_{kn}\|^2 p_{kn}^* = B_N \{max\{\frac{\|h_{kn}\|^2}{\lambda_k}\} - N_0\}$$

From (2), the sum-capacity on these r subchannels is

$$c_{r,sum} = \frac{B}{N} \sum_{n=1}^{r} \log_2 \left(\frac{max\{\frac{\|h_{kn}\|^2}{\lambda_k}\}}{N_0} \right) \tag{36}$$

Because $\max_k\{\frac{\|h_{kn}\|^2}{\lambda_k}\}$ and $r < K$ are finite, we know $c_{r,sum}$ is inverse proportion to N. If we assign each subchannel to only one user with the maximum weighted power gain, the capacity gap between OFDMA and the optimal solution is less than $c_{r,sum}$. From Equation (36), this gap is negligible when $N >> K$. Essentially, $c_{r,sum} \to 0$ when $N \to \infty$. Thus, OFDMA becomes the optimal solution when $N \to \infty$. ■

Remark 5: With some nonzero probabilities, the solution to KKT Conditions (14) to (18) requires multiple users to transmit over a finite number of subchannels. However, the capacity gap between OFDMA and the optimal solution vanishes when $N \to \infty$. This is exact the case in [3] and [18] where discrete multicarrier reduces to continuous frequency selective channels.

IV. Numerical Results

While Section III provides insights on OFDMA optimality in different regions, it is difficult to derive explicit analytical results for the general case (arbitrary SNR and N values). In this Section, we first present a fast numerical algorithm to compute the optimal solution to (4). We then calculate P_o by examining whether or not the optimal solution in each channel realization is OFDMA. Finally, we evaluate the performance of OFDMA by simulations.

For Gaussian vector multiple-access channels, Yu [5] proposed a fast iterative water filling method and proved it converges to the optimal point. A similar algorithm was proposed by

Tse and Hanly [4] where the optimal power allocation over time was characterized. Based on the same idea, we have the following algorithm for calculating the optimal sum-capacity in multicarrier uplink SISO channels

Algorithm 1: Iterative water-filling for multicarrier uplink SISO channels

initialize p_{kn}, $k = 1, ..., K$; $n = 1, ..., N$.

while (the desired accuracy is not reached)

for $k = 1$ *to* K

for $n = 1$ *to* N

$$Z_{kn} = I + \sum_{j=1,\, j\neq k}^{K} \|h_{jn}\|^2 p_{jn}$$

end

$$p_{kn} = \arg\max_{p_{kn}} \sum_{n=1}^{N} \log_2\left(Z_{kn} + \|h_{kn}\|^2 p_{kn}\right)$$

end

end

Corollary 2: Algorithm 1 converges to a limit point from any initial assignment of p_{kn}. *The limit point maximizes the sum-capacity of multicarrier uplink SISO channel in (4).*

Since multicarrier subchannels are parallel channels, the proof of Theorem 2.4 in [5] can be applied here by allowing single user water fill over frequency domain.

In practice, the parallel frequency selective subchannels are usually correlated. We consider subchannels with different correlation profiles in our simulations. The correlation among subchannels depends on the coherent bandwidth B_c and subcarrier spacing B_n. A general approximation for coherent bandwidth is $B_c \approx 1/T_m$, where T_m is typically taken to be the root mean square (rms) delay spread of the power delay profile. The power delay profile is often modeled as having a one-sided exponential distribution [16]:

$$A_c(\tau) = \tfrac{1}{T_m} e^{-\tau/T_m}, \quad \tau \geq 0 \qquad (37)$$

where the average and rms delay spread are the same as T_m.

The coherent bandwidth varies in different applications. For example, the subcarrier spacing in 802.11 a/g is 0.3125MHz for a 20MHz bandwidth with 64 subcarriers, which corresponds to a large B_c. In WiMax (IEEE 802.16e), the same bandwidth employes hundreds of subcarriers (from 128 up to 2048 depending on the application) because of relatively small B_c.

Assuming the power delay profile of (37), we consider two scenarios in the simulation: (1) weak correlation among subchannels, with $B_c = B_n$. (2) strong correlation among subchannels, with $B_c = 10B_n$. For both scenarios, we plot P_o and OFDMA capacity as a function of K, N and SNR. We assume power control is used so that the base station has the same received power for all users. In uplink scenario, users are usually separated far enough to be considered independent. For each user, we assume correlated Rayleigh fading among subchannels.

Figures 3 to 6 show P_o versus N for different Ks with different SNRs. We choose SNRs over a large range from -25dB to 25dB.

We can see from Fig. 3 that the numerical results match perfectly with our analysis in low SNR region (Theorem 2) for both weak and strong subchannel correlations. In both cases, P_o increases with N and decreases with K.

In Fig. 4, the P_o curves for $B_c = B_n$ match the low SNR analysis very well. This is even true for moderate SNR=-5dB (Figure is omitted here due to space limit). However, P_o decreases considerably for $B_c = 10B_n$. The intuitive explanation is that in low SNR region, each user tries to find the best subchannel over all subcarriers. But the strong correlation ($B_c = 10B_n$) among subchannels makes it difficult because adjacent subchannels have similar channel gains.

From Figures 5 and 6, we observe that the P_o curves for $B_c = B_n$ decrease in the moderate SNR region (5dB) and match with our analysis (Theorem 3) in high SNR region (15dB). We expect these results because subchannels are less correlated for $B_c = B_n$ and can be approximated as i.i.d. These are exactly the assumption we made for the derivation of Equation (29).

Interestingly, we observe that the P_o curves for $B_c = 10B_n$ agree with (29) for both SNR=5dB and SNR =15dB, which is especially true for large N. Note that in high SNR region each user is generally assigned multiple instead of only one subchannels. This presents the key difference from the low SNR region. Actually, even though adjacent subchannels are strongly correlated, a large N in high SNR region provides sufficient multiuser channel diversity compared to iid case.

We observe from Figures 3 to 6 that simulation results match our analyses in both low and high SNR regions regardless whether subchannels are correlated or not. In particular, our analyses for low and high SNR regions provide the upper and lower bounds for P_o in moderate SNR region respectively. In general, P_o increases with N and decreases with K and the SNR. Higher correlation among subchannels reduces P_o in moderate SNR region.

Next we illustrate the performance gap between OFDMA and the optimal solution. From Section III, we know that OFDMA is optimal when $N \to \infty$. If all users have the same individual power constraints, by symmetry, λ_ks are the same for all users. The optimal subchannel allocation policy is to allow only the user with the best channel gain $\|h_{kn}\|$ to transmit on subchannel n. We adopt this policy as the suboptimal subchannel allocation scheme for finite Ns and numerically evaluate its performance. Fig. 7 to Fig. 10 show the average OFDMA capacity versus N for different Ks. For each set of parameters (K, N and SNR), we normalize OFDMA capacity with respect to the maximum capacity, i.e., optimal sum-capacity is always 1.

Similar to P_o, we can see the normalized OFDMA capacity increases with N and decreases with K. Interestingly, even though P_o decreases with SNR, the normalized OFDMA capacity increases with SNR. Regardless of the SNR region, higher correlation degrades OFDMA capacity as expected since correlation reduces multiuser channel diversity. While OFDMA is not always optimal, it captures a majority of the same rate (80%) in all cases. Overall, OFDMA is nearly optimal in most practical situations, especially when N is large.

V. Conclusion

In this paper, we have investigated the sum-rate optimality of OFDMA in uplink multicarrier systems with a finite number of subchannels. We have derived the necessary and sufficient conditions under which OFDMA is sum-rate optimal and calculated the probabilities of these conditions for both low and high SNR regions. When OFDMA is suboptimal, we have shown the number of shared subchannels under the optimal solution must be less than the number of total users. While OFDMA is not optimal in general, the performance gap between the OFDMA scheme and the optimal solution is small in most cases. The difference diminishes when the number of subchannels is sufficiently large. Finally, we give a fast numerical algorithm to calculate the optimal solution and our simulations validate the analyses for both weak and strong subchannel correlations.

Appendix A: $-C_{sum}$ is a convex function

We shall prove the statement with the following Lemma [29].

Lemma 3: $f(x)$ is (strictly) convex if the Hessian of f is positive (definite) semi-definite.

We see from (4) that C_{sum} is simply the sum of the capacity on each subchannels. In order to prove $-C_{sum}$ is a convex function, we only need to prove the convexity of each summation term, i.e.,

$$-C_{sn} = -B_N \log_2 \left(1 + \frac{\sum_{k=1}^{K} p_{kn} \|h_{kn}\|^2}{B_N N_0} \right), \text{ where } 1 \leq n \leq N$$

For any subchannel n, we have

$$-\frac{\partial^2 C_{sn}}{\partial p_{jn} \partial p_{in}} = \frac{B_N \|h_{in}\|^2 \|h_{jn}\|^2}{\left(\sum_{k=1}^{K} \|h_{kn}\|^2 g_{kn} + B_N N_0 \right)^2} > 0, \text{ where } 1 \leq i, j \leq K.$$

Then the Hessian matrix of $-C_{sn}$ can be decomposed as:

$$\mathbf{S} = \frac{B_N}{\left(\sum_{k=1}^{K} \|h_{kn}\|^2 g_{kn} + B_N N_0 \right)^2} \mathbf{T}^T * \mathbf{T}$$

where $\mathbf{T} = \left[\|h_{1n}\|^2, \|h_{2n}\|^2, ..., \|h_{Kn}\|^2 \right]$. It is obvious that **S** is strictly positive definite. With Lemma 3, we state that $-C_{sn}$ is strictly convex and thus $-C_{sum}$ is strictly convex.

Appendix B: Proof of the first claim of Theorem 1

Proof: Note that Equation (14) can be rewritten as

$$\sum_{i=1}^{K} \|h_{in}\|^2 p_{in}^* = B_N \left(\frac{\|h_{kn}\|^2}{\lambda_k^* - \lambda_{kn}^*} - N_0 \right) \tag{38}$$

From Condition (18), we have

$$\sum_{i=1}^{K} \|h_{in}\|^2 p_{in}^* \geq B_N \left(\frac{\|h_{kn}\|^2}{\lambda_k^*} - N_0 \right) \tag{39}$$

If there exists at least one user k whose transmitting power $p_{kn}^* > 0$, we have $\lambda_{kn}^* = 0$ from (17). Thus Equation (38) becomes

$$\sum_{i=1}^{K} \|h_{in}\|^2 p_{in}^* = B_N \left(\frac{\|h_{kn}\|^2}{\lambda_k^*} - N_0 \right) \tag{40}$$

Since $\sum_{i=1}^{K} \|h_{in}\|^2 p_{in}^* > 0$, we have $\frac{\|h_{kn}\|^2}{\lambda_k^*} > N_0$. Thus, $\max_k \left\{ \frac{\|h_{kn}\|^2}{\lambda_k} \right\} > N_0$.

On the other hand, if $\frac{\|h_{kn}\|^2}{\lambda_k^*} > N_0$, from inequality (39), we obtain $\sum_{i=1}^{K} \|h_{in}\|^2 p_{in}^* > 0$, i.e., there exists at least one users with non-zero transmitting power on subchannel n.

Now we use reduction to absurdity to determine which user should transmit on subchannel n when $\max_k \left\{ \frac{\|h_{kn}\|^2}{\lambda_k} \right\} > N_0$. Assume user k_1 transmits ($p^*_{k_1 n} > 0$) with weighted power gain $\frac{\|h_{k_1 n}\|^2}{\lambda^*_{k_1}} < \max_k \left\{ \frac{\|h_{kn}\|^2}{\lambda_k} \right\}$. Equation (40) yields

$$\sum_{i=1}^{K} \|h_{in}\|^2 p^*_{in} = B_N \left(\frac{\|h_{k_1 n}\|^2}{\lambda^*_{k_1}} - N_0 \right) \tag{41}$$

We know, however, there exists a user k_2 with weighted power gain $\frac{\|h_{k_2 n}\|^2}{\lambda^*_{k_2}} > \frac{\|h_{k_1 n}\|^2}{\lambda^*_{k_1}}$. Equation (39) yields

$$\sum_{i=1}^{K} \|h_{in}\|^2 p^*_{in} \geq B_N \left(\frac{\|h_{k_2 n}\|^2}{\lambda^*_{k_2}} - N_0 \right) > B_N \left(\frac{\|h_{k_1 n}\|^2}{\lambda^*_{k_1}} - N_0 \right) \tag{42}$$

Equation (41) and (42) contradict with each other. Thus, we have proved that only those users with the maximum weighted power gain are allowed to transmit on subchannel n. ■

APPENDIX C: P_Ψ FOR HIGH SNR CASE

In this appendix, we derive P_Ψ, the probability of Condition (33), assuming $\|h_{kn}\|$s are i.i.d Rayleigh RVs.

For different subchannels m and n, since the event

$$A_1 = \left\{ \frac{\|h_{kn}\|}{\|h_{in}\|} \geq \sqrt{\frac{I_k}{I_i} \frac{p_i}{p_k}}, \forall\ i \neq k \right\} \tag{43}$$

is independent of the event

$$A_2 = \left\{ \frac{\|h_{jm}\|}{\|h_{im}\|} \geq \sqrt{\frac{I_j}{I_i} \frac{p_i}{p_j}}, \forall\ i \neq j \right\} \tag{44}$$

All we need is to derive $P\{A_1\}$. Via some random variable transformation, $P\{A_1\}$ becomes

$$\begin{aligned} & P\left\{ \|h_{kn}\| \geq \sqrt{\frac{I_k}{I_i} \frac{p_i}{p_k}} \|h_{in}\|, \forall\ i \neq k \right\} \\ &= P\left\{ \|h_{kn}\| \geq \max_{i,\ i\neq k} \left\{ \sqrt{\frac{I_k}{I_i} \frac{p_i}{p_k}} \|h_{in}\| \right\} \right\} \\ &= P\left\{ \max_{i,\ i\neq k} \left\{ \sqrt{\frac{I_k}{I_i} \frac{p_i}{p_k}} \|h_{in}\| \right\} - \|h_{kn}\| \leq 0 \right\} \end{aligned} \tag{45}$$

Let $X = \max\limits_{i,\, i \neq k} \left\{ \sqrt{\frac{I_k}{I_i}\frac{p_i}{p_k}} \|h_{in}\| \right\}$, $Y = -\|h_{kn}\|$ and $Z = X + Y$. Since $\|h_{kn}\|$ has Rayleigh distribution, using the monotonic transformation formula, we know immediately that the probability density function (PDF) of Y is

$$f_Y(t) = \begin{cases} -\frac{2}{b} t e^{-t^2/b} & t \leq 0 \\ 0, & t > 0 \end{cases} \tag{46}$$

where b is a constant. The cumulative distribution function (CDF) of $\sqrt{\frac{I_k}{I_i}\frac{p_i}{p_k}} \|h_{in}\|$ is thus

$$F(t) = \begin{cases} 1 - e^{-t^2 \frac{I_i p_i}{b I_k p_k}} & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{47}$$

Now we derive the distribution of X,

$$\begin{aligned} F_X(t) &= P\{X \leq t\} \\ &= P\left\{ \sqrt{\frac{I_k}{I_1}\frac{p_1}{p_k}} \|h_{1n}\| \leq t, ..., \sqrt{\frac{I_k}{I_{k-1}}\frac{p_{k-1}}{p_k}} \|h_{k-1,n}\| \leq t, \right. \\ &\quad \left. \sqrt{\frac{I_k}{I_{k+1}}\frac{p_{k+1}}{p_k}} \|h_{k+1,n}\| \leq t, ..., \sqrt{\frac{I_k}{I_K}\frac{p_K}{p_k}} \|h_{Kn}\| \leq t \right\} \\ &= \begin{cases} \prod\limits_{i \neq k} (1 - e^{-t^2 \frac{I_i p_i}{b I_k p_k}}) & t \geq 0 \\ 0, & t < 0 \end{cases} \end{aligned} \tag{48}$$

Since X and Y are independent, we have $F_Z(z) = F_X(t) * f_Y(t)$, where "$*$" means convolution. Then Equation (45) becomes

$$\begin{aligned} F_Z(0) &= \int_{-\infty}^{+\infty} F_X(t) f_Y(-t) dt \\ &= \int_0^{+\infty} \prod_{i \neq k} (1 - e^{-t^2 \frac{I_i p_i}{b I_k p_k}}) \frac{2}{b} t e^{-t^2/b} dt \end{aligned} \tag{49}$$

Let $x = e^{-t^2/b}$, (49) becomes

$$F_Z(0) = \int_0^1 \prod_{i \neq k} (1 - x^{\frac{I_i p_k}{I_k p_i}}) dx \tag{50}$$

Therefore

$$\begin{aligned} P\{\|h_{kn}\| &\geq \sqrt{\frac{I_k}{I_i}\frac{p_i}{p_k}} \|h_{in}\| \text{ for } \forall\ i \neq k\} \\ &= \int_0^1 \prod_{i \neq k} (1 - x^{\frac{I_i p_k}{I_k p_i}}) dx \end{aligned} \tag{51}$$

Since parallel subchannels are independent, the probability of Condition (33) is given by

$$P_\Psi = [\int_0^1 \prod_{j\neq 1}(1 - x^{\frac{I_j p_1}{I_1 p_j}})dx]^{I_1} ... [\int_0^1 \prod_{j\neq K}(1 - x^{\frac{I_j p_K}{I_K p_j}})dx]^{I_K} \tag{52}$$

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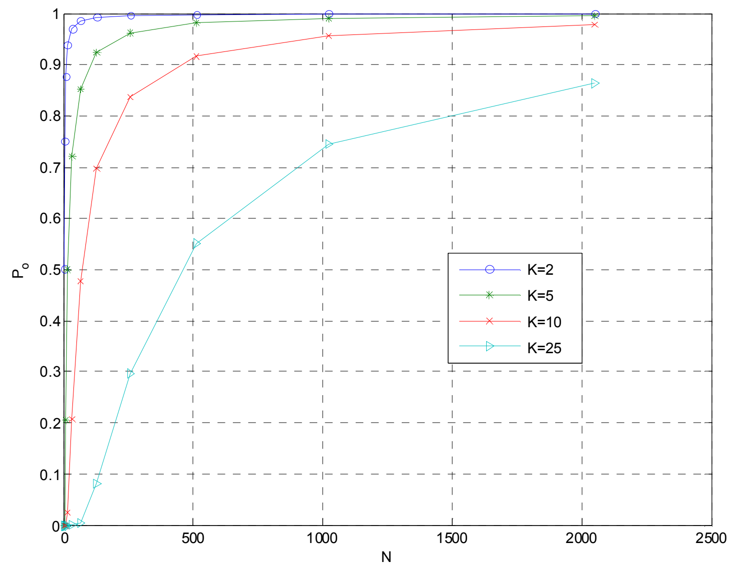

Fig. 1. Po for Low SNR

Conditions	Power loading/subchannel allocation	OFDMA Optimal?
$\alpha_1 \leq \alpha_2$ $\lvert \alpha_2 p_1 + n_{22} - n_{21} \rvert < p_2$	$p_{11} = 0,$ $p_{21} = \frac{1}{2}(\alpha_2 p_1 + p_2 + n_{22} - n_{21}),$ $p_{12} = p_1,$ $p_{22} = \frac{1}{2}(-\alpha_2 p_1 + p_2 + n_{21} - n_{22}),$	No
$\alpha_2 p_1 + n_{22} - n_{21} \geq p_2$ $\frac{p_2}{\alpha_1} + n_{11} - n_{12} \geq p_1$	$p_{11} = 0, \quad p_{12} = p_1,$ $p_{21} = p_2, \quad p_{22} = 0,$	Yes

Table 1 Example 1

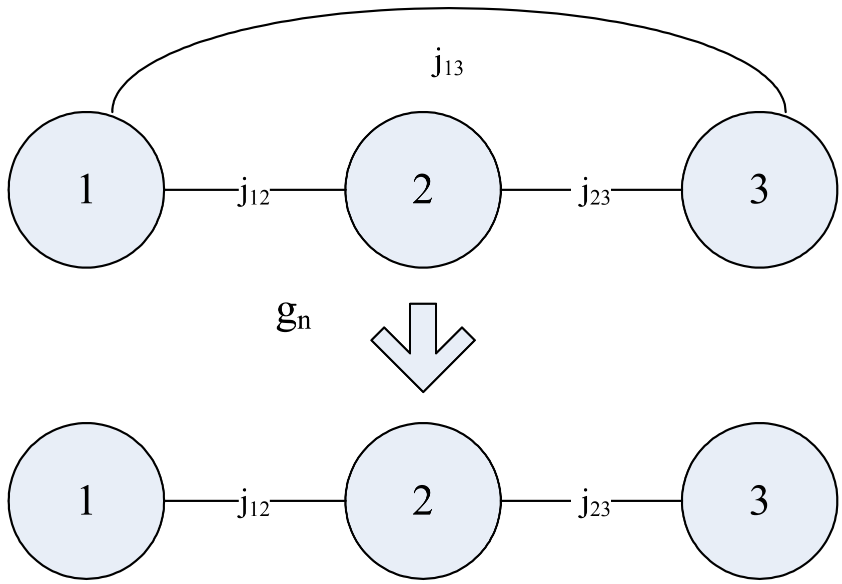

Fig. 2. Example 2

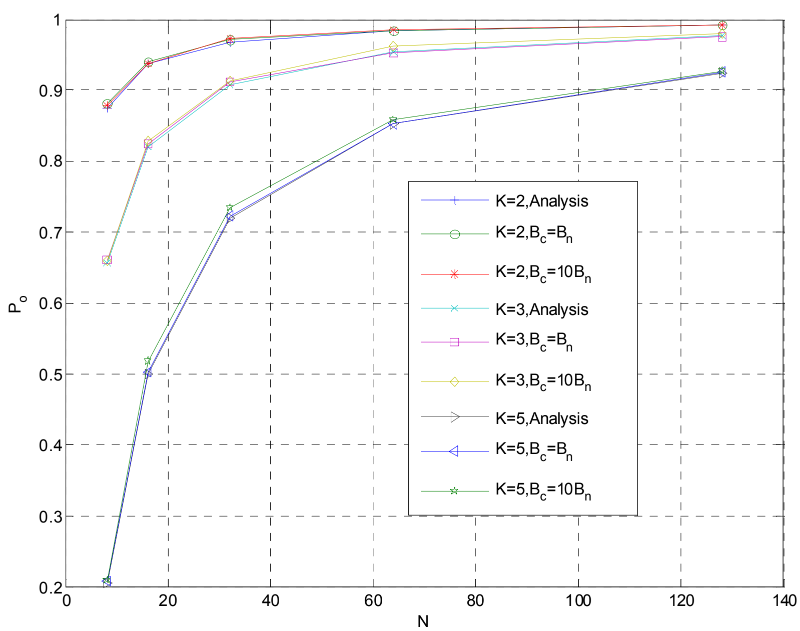

Fig. 3. P_o for SNR=-25dB

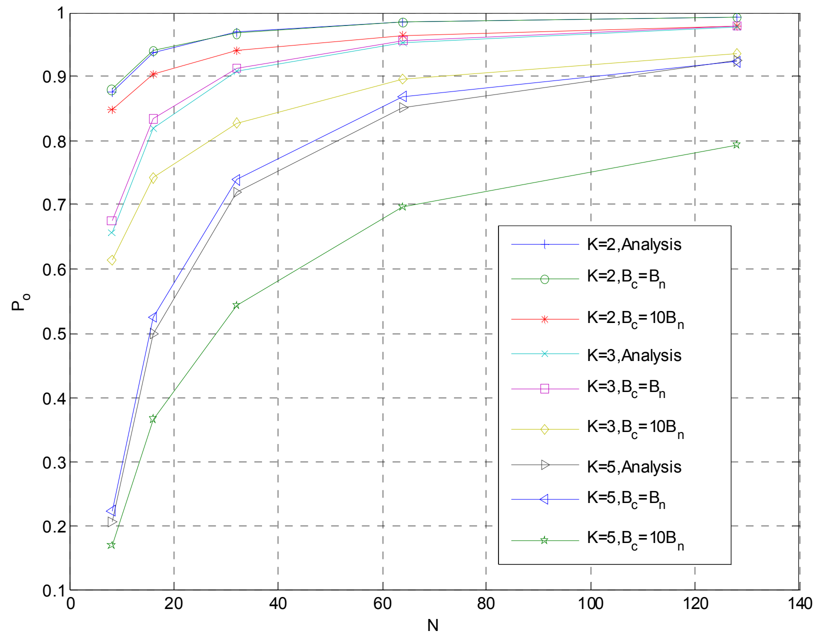

Fig. 4. P_o for SNR=-15dB

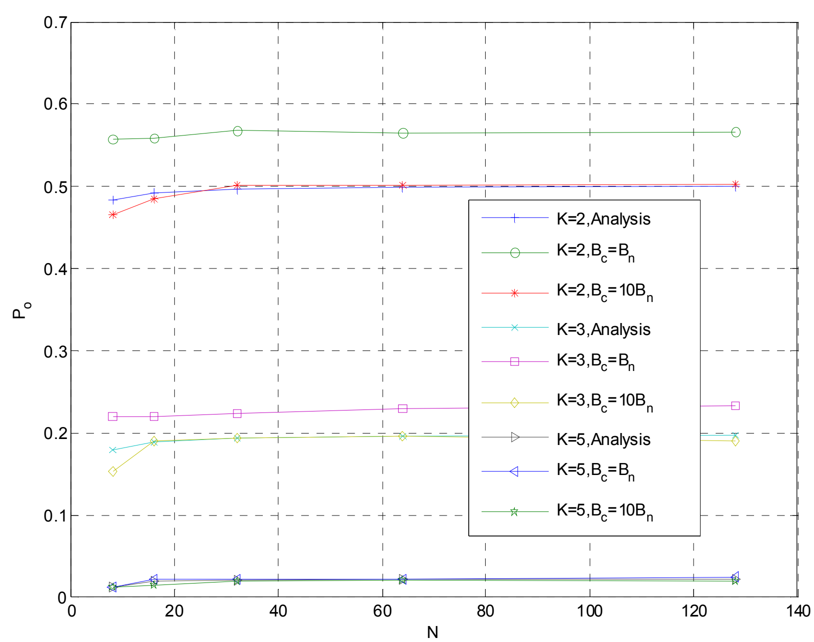

Fig. 5. P_o for SNR=5dB

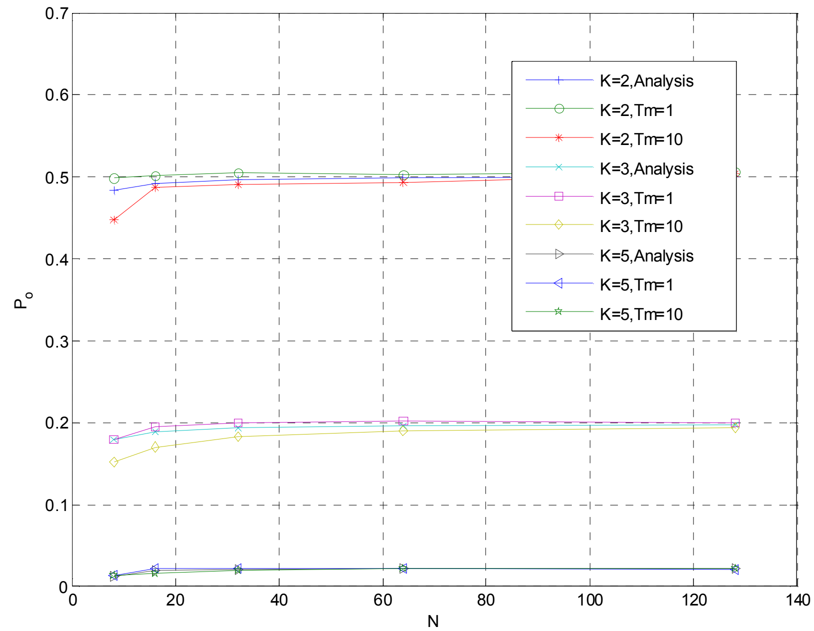

Fig. 6. P_o for SNR=15dB

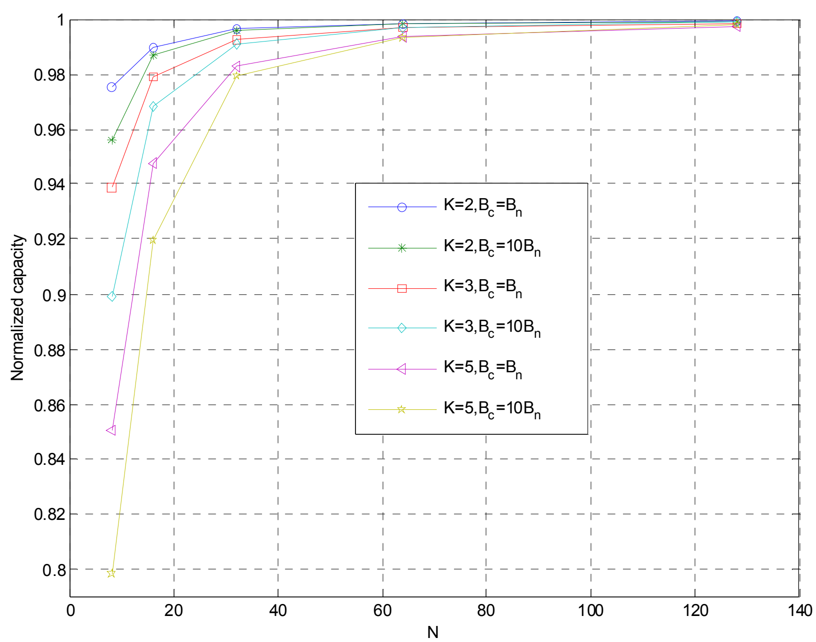

Fig. 7. Normalized OFDMA capacity, SNR=-25dB

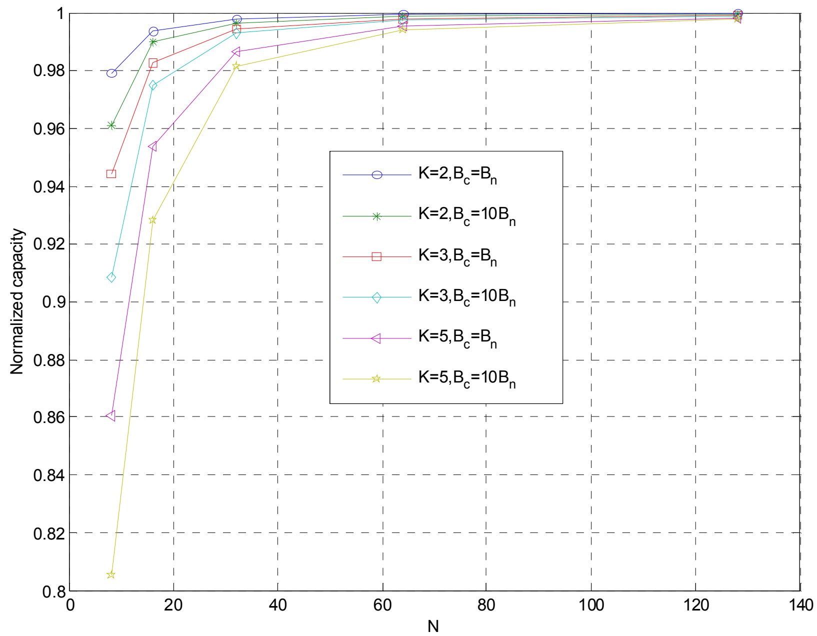

Fig. 8. Normalized OFDMA capacity, SNR=-15dB

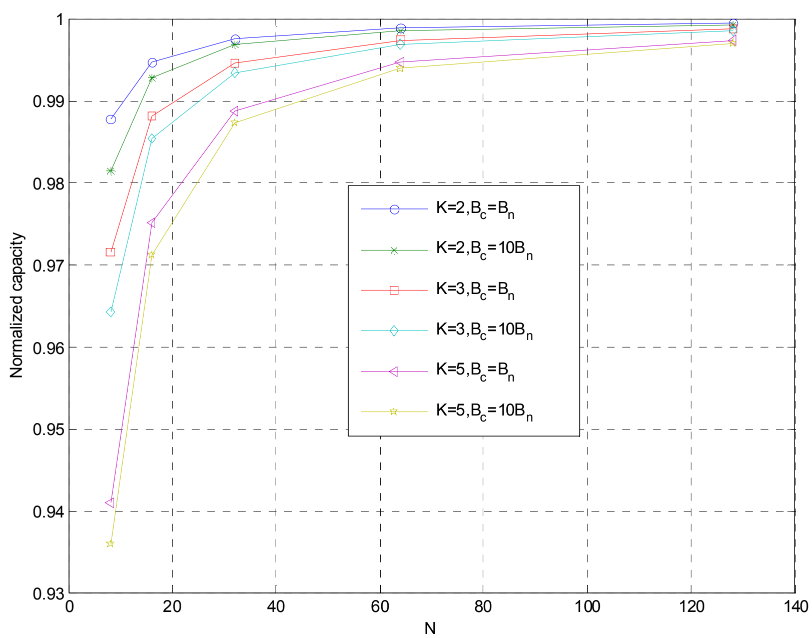

Fig. 9. Normalized OFDMA capacity, SNR=5dB

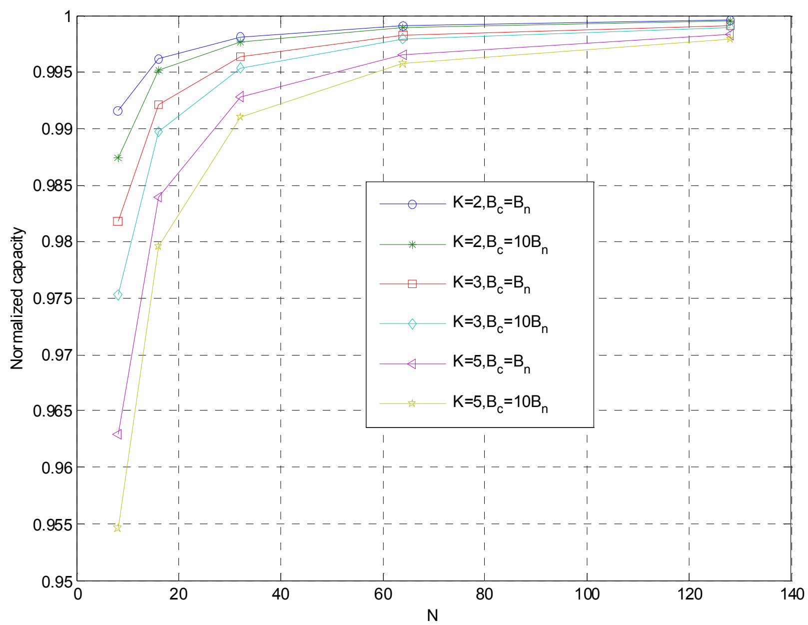

Fig. 10. Normalized OFDMA capacity, SNR=15dB